1. (20 Points) Answer whether the followings are true and prove your answer:

(a) \( \sqrt[n]{n} \) is \( O(\sqrt[n]{n} + \log(n)) \)

Solution
No. (2 pts)
Proof (2 pts) Suppose that \( \sqrt[n]{n} \) is \( O(\sqrt[n]{n} + \log(n)) \). By the supposition that \( \sqrt[n]{n} \) is \( O(\sqrt[n]{n} + \log(n)) \), there exists a positive real number \( B \) and a real number \( b \) such that for all integers \( n > b \)
\[
|\sqrt[n]{n}| \leq B|\sqrt[n]{n} + \log(n)|
\]  
(1)

Let \( n \) be a positive integer that is greater than \( \max\{(2B)^{12}, 2^{16}, b\} \). Then
\[
n > (2B)^{12} \implies |\sqrt[n]{n}| = \sqrt[12]{n^{12}} \sqrt[n]{n} > 2B \sqrt[n]{n}
\]
\[
\sqrt[n]{n} > \log(n) \text{ for all integers } n > 2^{16} \implies 2B \sqrt[n]{n} > B(\sqrt[n]{n} + \log(n))
\]
\[
\therefore |\sqrt[n]{n}| > B(\sqrt[n]{n} + \log(n)) = B|\sqrt[n]{n} + \log(n)|.
\]

Thus there is an integer \( n > \max\{(2B)^{12}, 2^{16}, b\} \geq b \) such that
\[
|\sqrt[n]{n}| > B|\sqrt[n]{n} + \log(n)|.
\]

This contradicts (1). Hence the supposition is false, and so \( \sqrt[n]{n} \) is not \( O(\sqrt[n]{n} + \log(n)) \).

(b) \( \sqrt[100]{n} \) is \( O(\log n) \)

Solution
No. (2 pts)
Proof (2 pts) Suppose that \( \sqrt[100]{n} \) is \( O(\log n) \). By the supposition that \( \sqrt[100]{n} \) is \( O(\log n) \), there exists a positive real number \( B \) and a real number \( b \) such that for all integers \( n > b \)
\[
|\sqrt[100]{n}| \leq B|\log n|
\]  
(2)

Let \( B' = \max\{B, 1\} \). Let \( n \) be a positive integer that is greater than \( \max\{2^{1000B'}, b\} \). Then

We have \( \sqrt[100]{n} > B' \log n \geq B \log n \).
\[
\therefore |\sqrt[100]{n}| > B|\log n|.
\]

Thus there is an integer \( n > \max\{2^{1000B'}, b\} \geq b \) such that
\[
|\sqrt[100]{n}| > B|\log n|.
\]

This contradicts (2). Hence the supposition is false, and so \( \sqrt[100]{n} \) is not \( O(\log n) \).

(c) \( \log^3 n \) is \( O(\log n) \)

Solution
No. (2 pts)
Proof (2 pts) Suppose that \( \log^3 n \) is \( O(\log n) \). By the supposition that \( \log^3 n \) is \( O(\log n) \), there exists a positive real number \( B \) and a real number \( b \) such that for all integers \( n > b \)
\[
|\log^3 n| \leq |\log n|
\]  
(3)
Let $n$ be a positive integer that is greater than $\max\{2^{B^2}, b\}$. Then

$$\log^3 n = \log^2 n \log n > B \log n.$$

Thus there is an integer $n > b$ such that

$$|\log^3 n| > B |\log n|.$$

This contradicts (3). Hence the supposition is false, and so $\log n$ is not $O(\log n)$.

(d) Let $a, b$ be any positive integers.

$(a + n)^b$ is $O(n^a)$

Solution

No. (2 pts)

Proof (2 pts) Let $a = 1, b = 2$. We show $(1 + n)^2$ is not $O(n)$. This can be proved by Theorem 9.2.2. Hence, $(a + n)^b$ is $O(n^a)$ does not hold for all $a, b$ that are positive integers.

(e) $|\sqrt{n}|$ is $O(\sqrt{n})$.

Solution

Yes. (2 pts)

Proof (2 pts) For any integer $n > 1$, $||\sqrt{n}|| = |\sqrt{n}|$ because since $n > 1 > 0$, then $|\sqrt{n}| > 0$

$$\Rightarrow |\sqrt{n}| \leq \sqrt{n} \text{ because } ||x|| \leq x \text{ for all real numbers } x$$

$$\Rightarrow |\sqrt{n}| \leq \sqrt{n} \text{ because } \sqrt{n} \geq 0.$$

Let $M = 1$ and $x_0 = 1$. Then $||\sqrt{n}|| \leq M|\sqrt{n}|$ for all integers $n > x_0$, and so by definition of $O$-notation, $|\sqrt{n}|$ is $O(\sqrt{n})$.

2. (10 Points) A label has the following format: Letter Letter Digit Digit Letter Letter. (Digit is non-zero and we use capital Letter only.)

(a) (2.5 pts) How many different labels are there?

$26^2 \times 9^2 \times 26^2$

(b) (2.5 pts) How many different labels are there with distinct letters and digits?

$26 \times 25 \times 9 \times 8 \times 24 \times 23$

(c) (2.5 pts) How many different labels could begin with $AB$ and have non-repeating letters or digits?

$9 \times 8 \times 24 \times 23$

(d) (2.5 pts) How many different labels have the letters $A$ and $B$ and have non-repeating letters or digits?

$P(4, 2) \times 9 \times 8 \times 24 \times 23$ [In the first step we find the positions for $A$ and $B$ (there are $P(4, 2)$ ways). Then we choose the remaining letters and digits.]

3. (20 Points) There are 6 students named A,B,C,D,E,F respectively. Now they need to stand in a line. How many are the different ways to form the line

(a) if $A$ is neither the first nor the last (6th) one.

(b) $A$ and $B$ are either the first or the last (6th) one.
(c) A and B must stand adjacent to each other.
(d) A and B must not stand adjacent to each other.

Solution

(a) (5 pts) In the first step, A can stand in any position except 1 and 6, so A has 4 ways to choose. In the second step, the other 5 students will choose the remaining 5 positions after A has chosen his position, so totally: \(4 \times P(5, 5) = 4 \times 5! = 480\).

(b) (5 pts) In the first step, A and B must stand at either 1 or 6 position (there are only two cases: A stands first and B stands last; A stands last and B stands first), so these two students have \(P(2, 2)\) ways to arrange themselves. In the second step, the other 4 students will choose the remaining 4 positions after A and B have chosen their positions, so totally: \(P(2, 2) \times P(4, 4) = 48\).

**Remark** If you interpret “A and B are either the first or the last (6th) one” as “A is the first or A is the last or B is the first or B is the last”, then the answer should be \(P(6, 6) - P(4, 2) \times P(4, 4) = 720 - 12 \times 24 = 432\). \(P(4, 2) \times P(4, 4)\) counts for the number of ways that neither the first nor the last position is taken by A or B. So in the first step we pick up two positions from the positions 2,3,4,5 for A and B (there are \(P(4, 2)\) ways); then in the second step C,D,E,F will be arranged to the remaining four positions (there are \(P(4, 4)\) ways). So for this problem, we have \(P(6, 6) - P(4, 2)P(4, 4)\) ways.] This will also get full points.

(c) (5 pts) We can consider A and B as a small group, in this group, A can either stand in front of B or behind B, so this small group has 2 ways to arrange themselves. Then C,D,E,F and AB can be considered as a big group containing 5 elements for permutation, thus there are \(P(5, 5)\) ways. Totally, \(2 \times P(5, 5) = 240\).

(d) (5 pts) In Problem 3(c), we have counted the number of ways which require A and B to stand together. So for this problem, we have \(P(6, 6) - 240 = 480\) ways.

4. (10 Points) One day you go hiking with your friends. In case you are too far apart to hear from each other, each one get three flags with different colors: red, yellow and blue respectively. Then you want to use these flags as special signs (For example, if you raise yellow flag at first, then red flag, and finally blue flag, it means: I need help.). You can use one of the flags, two flags or three flags with different sequences to represent different signs. How many different signs can be represented totally by these three flags?

**Solution**
\[P(3, 1) + P(3, 2) + P(3, 3) = 15\]

**Remark** We explained on the class webpage that *each sequence that represents a sign does not contain repeating flags with the same color*. In case you did not read this note and consider sequences with repeating flags with the same color, the answer should be \(3 + 3 \times 3 + 3 \times 3 \times 3 = 39\).
This will also get full points.