Final Exam, CSE21, Fall 2002

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<th>Problem</th>
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There are 100 points and you have 180 minutes. However, be careful about timing: Some problems are relatively straightforward and will quickly give you a good number of points, while others are relatively difficult and will require deep thinking. Do not waste too much time on the hard ones before you secure the easy points.

You may not use books and notes. You may use calculators.

**DO NOT START UNTIL WE TELL YOU SO**

Good Luck!
Problem 1 35 points

This is a multiple choice question. You do not need to justify your answers. You must circle the number corresponding to the correct choice in each case. Points are awarded like this:

- Each correct answer is worth five points.
- Leaving it blank gets you zero points.
- For a wrong answer two points will be deducted.

(1) Consider a universal set $U$ and two events $A$ and $B$, such that $A \cap B = \emptyset$, $P(A) > 0$ and $P(B) > 0$. Then

1. $P(B|A) = P(B)$
2. $P(B|A) = P(B)P(A)$
3. $P(B|A) = 0$
4. we cannot determine from the above
5. none of the above

(2) Consider a universal set $U$ and two events $A$ and $B$, such that $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$ and $A$ and $B$ are independent. Then $P(A \cup B) = \ldots$

1. $\frac{1}{3}$
2. $\frac{1}{2}$
3. $\frac{7}{12}$
4. we cannot determine from the above
5. none of the above

(3) Consider a universal set $U$ and three events $A$, $B$, and $C$ such that $A$, $B$ and $C$ are independent and $P(A) > 0$, $P(B) > 0$ and $P(C) > 0$. Then

1. $P(A|B \cap C) = P(A|C)$
2. $P(A \cup B \cup C) = P(A)+P(B)−P(A)P(B)+P(C)−[P(A)+P(B)−P(A)P(B)]P(C)$
3. all of the above
4. none of the above
5. we cannot determine from the given information
(4) Let $A$ and $B$ be events with $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$ and $P(A \cup B) = \frac{7}{12}$. Then $P(A|B) =$

1. $\frac{1}{6}$
2. $\frac{1}{12}$
3. 0
4. we cannot determine from the above
5. none of the above

(5) Again, $A$ and $B$ are events with $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$ and $P(A \cup B) = \frac{7}{12}$. Now we consider $B^c = U - B$, where $U$ is the universal set. Then $P(A|B^c) =$

1. $\frac{4}{9}$
2. $\frac{7}{9}$
3. $\frac{5}{9}$
4. cannot be determined from the given information
5. none of the above

$^1 B^c$ is called the complement of $B$. 
(6) Consider a simple graph $G$ and a walk $w$ in $G$. Which of the following statements is true?

1. If the walk contains a repeated vertex then the graph has a simple circuit, i.e., a circuit with no repeated edges and no repeated vertices;
2. If the walk contains every vertex of the graph exactly twice and uses all edges then the graph has an Euler circuit;
3. If the walk contains every edge of the graph then the graph is connected;
4. all of the above;
5. none of the above

(7) Consider a simple graph $G$ with $n$ nodes and $n - 1$ edges. Which of the following statements are true?

1. If the graph is connected then it is a tree.
2. If the graph has no circuit then it is connected.
3. If the graph is connected then it has no circuit.
4. Some but not all of 1, 2, 3.
5. All of 1, 2, 3.
6. None of 1, 2, 3.
Problem 2 6 points A school has 12 students and offers three classes: CSE101, CSE102, and CSE103. Only four students can enroll in each class. Each student can enroll in only one class. In how many ways can the 12 students take the 3 classes? Justify your answer.
Problem 3 8 points Two cards are drawn at random from an ordinary deck of 52 cards without replacement. Find the probabilities that

1. both are spades

2. one is a spade and one is a heart

Justify your answers.
Problem 4 12 points Let three fair coins be tossed. The universal set $U$ contains the following eight outcomes

$$HHH, HHT, HTH, HTT, THH, THT, TTH, TTT$$

Consider the following three events $A$, $B$, and $C$:

1. The event $A$ consists of outcomes with all heads or all tails, i.e., $A = \{HHH, TTT\}$;
2. The event $B$ consists of outcomes with at least two heads;
3. The event $C$ consists of outcomes with at most two heads;

Answer the following questions. Prove your answer.

1. Are $A$ and $B$ independent?
2. Are $A$ and $C$ independent?
3. Are $B$ and $C$ independent?
Problem 5  22 points A fair die is tossed. Let $X$ be the random variable that is twice the number appearing and let $Y$ be 1 if an odd number appears or 3 if an even number appears. For example, $X(5) = 10$ and $Y(5) = 1$. Answer the following. Show the steps you took towards deriving the solutions.

1. Find the probability distribution $f_X$;

2. Find the probability distribution $f_Y$;

3. Find the expectation $E(X)$;

4. Find the expectation $E(Y)$;

5. Find the variance $Var(X)$;

6. Find the expectation $E(X + Y)$;

7. Find the expectation $E(XY)$. 

Problem 6 16 points A sequence is defined recursively as follows

\[ t_k = k - t_{k-1}, \quad k > 0 \]
\[ t_0 = 0 \]

1. Guess an explicit formula for the sequence. You do not have to explain how you guessed what you guessed.

2. Use induction to prove that the formula that you guessed in (1) is correct.
Problem 7 16 points Consider the sequence $a_0, a_1, a_2, \ldots$ defined by the following recurrence relation.

\[
\begin{align*}
    a_k &= 2a_{k-1} - a_{k-2}, k > 1 \\
    a_0 &= 1 \\
    a_1 &= 4
\end{align*}
\]

Find an explicit formula for the sequence. Show the steps that you follow in order to derive the answer.
Problem 8 20 points When Alice and Bob play a game of backgammon in the morning, Alice has 60% probability to win. (A backgammon game cannot end up tied.) When they play in the evening Alice has only 40% probability to win. We know that Alice and Bob played a series of 20 games in a row, between 7:00 and 10:00 but we do not know if it was 7:00AM-10:00AM (morning games) or 7:00PM-10:00PM (evening games). Find the probability that the games were played in the morning given the information that Alice ended up winning 11 of the 20 games. Show your work.

*Hint:* Use the normal distribution approximation of the binomial distribution.
Problem 9 20 points Box A contains nine cards numbered 1 through 9, and box B contains five cards numbered 1 through 5. A box is chosen at random and a card is drawn from the chosen box. If the card shows an even number, another card is drawn from the same box. If the card shows an odd number, a card is drawn from the other box.

1. What is the probability that both cards show even numbers? Justify your answer.

2. If both cards show even numbers, what is the probability that they come from box A? Justify your answer.
Problem 10 15 points Consider the set of nodes $V = \{a, b, c, d, e, f\}$.

1. How many simple directed graphs can you build using $V$? Figure 1 shows an example directed graph. Its set of edges is

$$E = \{(a, f), (b, a), (b, b), (b, d), (b, e), (c, e), (e, d), (e, f), (f, a)\}$$

2. How many simple undirected graphs can you build using $V$? Figure 2 shows an example undirected graph. Its set of edges is

$$E = \{\{a, f\}, \{b, a\}, \{b\}, \{b, d\}, \{b, e\}, \{c, e\}, \{e, d\}, \{e, f\}\}$$

Justify your answers.
Figure 2: An example of simple undirected graph with set of nodes $V$

**Problem 11 10 points**

1. Does the following graph have an Euler circuit? If yes, write the Euler circuit. If not, prove why not.

2. Does it have a Hamiltonian circuit? If yes, write the Hamiltonian circuit. If not, prove why not.