CSE 21 Homework 3 Solutions

1. These two subproblems deal with functions.

a. This problem asks us to determine the number of functions and injections that map $A$ to $B$.

(A total of 6 points for Part A)

i. Since a function must map every element in the domain ($A$) to exactly one element in the range ($B$), a function can be thought of a $k$-list with repetition allowed. That is, for each element in $A$, we need to choose an element to $B$ that it maps to. At each choice we can choose from one of $|B|$ elements. Hence, there is precisely $|B|^{|A|}$ functions, or $5^3 = 125$.

(Worth 3 points)

ii. Recall that with an injection, each value in the domain must map to a unique value in the range. That is, no two distinct elements in the domain can map to the same value in the range. Therefore, we can think of an injection as the number of $k$-lists without repetition, which is the number of permutations where $n = |B|$ and $k = |A|$. We know this value to be $\frac{|B|!}{(|B|-|A|)!}$, or $\frac{5!}{(5-3)!}$, or $5 \cdot 4 \cdot 3$, or 60.

(Worth 3 points)

b. Given $f(x) = |x|$ and $f : \mathbb{R} \to \mathbb{R}$, $f^{-1}$ is not a function since $f$ is not a bijection. (Note that $f$ is neither a surjection or injection, hence it can't be a bijection...) Specifically, $f^{-1}$ cannot be a function since there exist elements in its domain that map to multiple values in the range. For example, the domain element 1 and $-1$ both map to the range value 1, thereby violating the definition of a function. Since $f^{-1}$ is not a function, it is neither a surjection, injection, or bijection.

(Part B is worth 4 points)

2. a. (4 points, 2 for $f \circ g$, 2 for $g \circ f$)

$f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 4 & 5 & 1 & 3 & 2 & 7 \end{pmatrix}$

$g \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 4 & 3 & 2 & 6 & 1 & 5 \end{pmatrix}$

b. (2 points)

$f^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 4 & 3 & 7 & 2 & 6 & 1 \end{pmatrix}$

$g^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 7 & 4 & 5 & 6 & 2 \end{pmatrix}$

So, $f^{-1} \circ g^2 = (5, 4, 3, 7, 2, 6, 1)$. 

1
c. (2 points)
First Step: choose $1 \times 2 = 2$ elements from 7 elements to form 1 2-cycle. That is, \( \binom{7}{2} = 21 \).
Second Step: There is only one arrangement of 2-size block.
Third Step: Using Rule of Product, $21 \times 1 = 21$.
Or
Use formula on the book, \( \frac{n!}{(n-2k)!2^k k!} = \frac{7!}{(7-1 \times 2)!2^2 1!} = 21 \).

d. (2 points)
Involutions with at least two 2-cycles on elements include involutions with two 2-cycles, and involutions with 3 2-cycles.
Now we compute the number of involutions with three 2-cycles.
First Step: choose $3 \times 2 = 6$ elements from 7 elements to form 3 2-cycles. That is, \( \binom{6}{2,2,2} = 7 \).
Second Step: Partition the 6 elements into 3 blocks that are all of size 2. That is, \( \frac{1}{3!} \times \binom{6}{2,2,2} = 15 \).
Third Step: Using Rule of Product, $7 \times 15 = 105$.
Or
Use formula on the book, \( \frac{n!}{(n-2k)!2^k k!} = \frac{7!}{(7-3 \times 2)!2^3 3!} = 105 \).
We use the same method to compute the number of involutions with two 2-cycles.
\( \frac{n!}{(n-2k)!2^k k!} = \frac{7!}{(7-2 \times 2)!2^2 2!} = 105 \).
At last, use Rule of Sum, $105 + 105 = 210$.

3. **#3**

(10 points)

Part A: (3 points)

A permutation that requires a power of 4 as the smallest to obtain the identity function cannot have a cycle bigger than 4 or a cycle of three. Once such permutation from the set \( \mathbb{Z}_7 \) is \( f = (1, 3, 4, 6)(2, 7)(5) \).
1 point for proper cycle notation/correct permutation, 1 point for a 4-cycle, and 1 point for no cycles of length 3 or greater than 4.

Part B: (3 points)
A permutation that is an involution has cycles only of length 1 or 2. Two such involutions from the set $\mathbb{S}$ are $f = (1, 2)(3, 5)(4)$ and $f = (1)(2)(3)(4)(5)$.

1 point for a function that’s a permutation, 1 point for proper cycle notation, and 1 point for all cycles having length 1 or 2.

Part C: (4 points)

There is no such thing as a permutation with no cycles. In order to explain this, one of two things must be shown: that if a function has no cycles, it is not a permutation, or that if it’s a permutation, it must have cycles.

Either argument uses these facts: Assume a permutation has no cycles. By definition, a permutation is a bijection that maps the domain to itself. Also by definition, a cycle is a list of mappings where the range element in one mapping is the domain element of the next and the last range element is the first domain element (if the cycle has only one mapping, only the second rule applies). The argument should look something like this:

Assume permutation $f$ has no cycles. Choose some element of the domain of $f$ (where the domain has $n$ elements) and call it $i_1$. Thus $f(i_1) \to i_2$, where $i_2$ is also in the domain of $f$. $f(i_2) \to i_3$, where $i_3$ is also in the domain of $f$, and we can continue to create a list of mappings this way through $f(i_{n-1}) \to i_n$. In order for there not to have been any cycles in this list of mappings, the elements $i_1$ through $i_n$ must all have been distinct values of the domain/range. In fact we could create such a list of mappings because we know the range element produced by a given domain element in the list must be different than all the domain elements that have previously been mapped (because there are no cycles).

However, there is one more mapping required to complete the specification of this permutation with no cycles: $f(i_n)$. Since $f$ is a bijection, the value of $f(i_n)$ must be different from any of the other range values $i_2$ through $i_n$. Since there are only $n$ elements to choose from, there is only one element $f(i_n)$ could map to: $i_1$. Therefore, $f(i_n) \to i_1$. However, this creates a cycle of length $n$, since we have created a list of mappings where the range element of one mapping is the domain element of the next, and the last range element is the first domain element. Thus, if $f$ is a permutation, it must have at least one cycle.

The last paragraph can also use the argument that if $f$ is to contain no cycles, it must map $f(i_n)$ to something other than $i_1..i_n$ since all of those are already listed as domain elements and would thus create a cycle. $f(i_n)$ must instead map to some other element $i_{n+1}$. However,
since the domain of \( f \) has only \( n \) elements, \( f \) cannot be a permutation. Thus if \( f \) is a permutation, it must have at least one cycle.

2 points for using the definitions of permutation and cycle, 1 point for use of a successful argument (I'd be surprised to see any successful arguments but the above two), and 1 point for clarity (if you can easily follow the student’s logic, give them the point. If you have to work hard to understand the point they are making and the arguments that support it, don’t give it).

4. **#4**

(10 points)

As the hint says, define \( J \) to be a random variable that is the amount won or lost in one round of playing the game. Thus the problem becomes “find the expected value of \( J \)”. The expected value of \( J \), or \( E(J) \), is equal to \( \Sigma(J_i \cdot P(i)) \) for all \( i \) where \( i \) is an event (in this case, there are three plus one not mentioned in the problem).

The probabilities of each case are:

\[
P(\text{both are } \leq 4) = \frac{4 \cdot 4}{6 \cdot 6} = \frac{16}{36} = \frac{4}{9}.
\]

\[
P(\text{at least one } = 5) = \frac{6 + 5}{36} = \frac{11}{36}.
\]

\[
P(\text{both are } 6) = \frac{1}{36}.
\]

You’ll notice the probabilities in this circumstance only add up to \( (16 + 11 + 1)/36 = 28/36 \). What’s left? Unmentioned is the case where exactly one of the dice is 6 but the other is not 5, which has probability \( 8/36 = 2/9 \) (11 ways where at least one is 6 minus the case where both are six minus the two cases where one is five). Since this case isn’t mentioned, we’ll assume there’s no gain or loss. Since \( J = 0 \) in this case, we could leave it out of our calculation because \( 0 \cdot P(J = 0) \) is equal to 0.

So explicitly, \( E(J) = \Sigma(J_i \cdot P(i)) = (-1 \cdot 4/9) + (2 \cdot 11/36) + (3 \cdot 1/36) + (0 \cdot 2/9) = 0.25 \), so the expected winnings each round is $0.25.
It’s also semi-reasonable to have assumed that rolling exactly one 6 without the other being
5 wins $2 because it might be considered better than rolling one 5. In this case the answer
is \((-1 \cdot 4/9) + (2 \cdot (11/36 + 8/36)) + (3 \cdot 1/36) = 0.694\), for an expected winning of $0.69 per
round.

1 point for each probability specified (not including the unmentioned one), 1 point for speci-
fying the random variable, 3 points for setting up the equation correctly, and 2 points for one
of the two answers. 1 point for noticing the unmentioned result and stating an assumption
about it (if they never mention it and get the first calculated answer correctly, take off a point
because they appeared not to consider how that might have affected the expected value).

5. \(E(X)\) (3 points) The probability of choosing one of the three boxes is \(\frac{1}{3}\).
So, \(E(X) = \frac{1}{3} \times 10 + \frac{1}{3} \times 20 + \frac{1}{3} \times 30 = 20\).

\(E(Y)\) (3 points) The probability of choosing one of the ball is \(\frac{1}{60}\).
The chosen number falls in
\[
\begin{align*}
& [1,10] \quad Y = 10 \\
& [11,30] \quad Y = 20 \\
& [31,60] \quad Y = 30
\end{align*}
\]
So, \(E(Y) = \frac{1}{60} \times 10 \times 10 + \frac{1}{60} \times 20 \times 20 + \frac{1}{60} \times 30 \times 30 = \frac{70}{3} \approx 23.33\)

Explanation (4 points) Although both \(X\) and \(Y\) denote the number of balls in a box, the sample space
we choose from in case \(X\) is the set of boxes, while in case \(Y\) the sample space is the set
of balls.

When the sample space is the set of boxes, each box has the same probability \(\frac{1}{3}\) of being
selected, so the mean of \(X\), which is \(E(X)\), is 20.

But when the sample space is the set of balls, each box has different probability of being
selected, the first box has the lowest probability \((\frac{1}{5})\) of being chosen, while the third
box has the highest probability \((\frac{1}{2})\) of being chosen, since it contains most balls. So the
mean of \(Y\) is greater than 20.

6. Recall that a random variable is a function defined as follows: \(X : U \rightarrow \mathbb{R}\). To begin this
problem we must first define \(U\), our universal set. \(U\) is the number of ways we can select a
committee of 3 from a pool of 10 people. Convince yourself that \(|U| = \binom{10}{3}\). The image of \(X\)
is \{0,1,2,3\}, since a committee of 3 people may have zero, one, two, or three males in it.

Now, to compute \(f_X\), we must iterate through the \(image(X)\) and for each element \(r \in \image(X),\) we compute \(f_X(r)\), which equals \(P(X^{-1}(r))\). So, we need to compute \(f_X(0), f_X(1), f_X(2),\) and \(f_X(3)\).

\[
f_X(0) = \frac{\binom{4}{3} \cdot \binom{6}{0}}{\binom{10}{3}} = \frac{4}{120} = 0.03333...
\]
\[ f_X(1) = \frac{\binom{4}{1} \cdot \binom{6}{3}}{\binom{10}{3}} = \frac{36}{120} = 0.3 \]

\[ f_X(2) = \frac{\binom{4}{2} \cdot \binom{6}{2}}{\binom{10}{3}} = \frac{60}{120} = 0.5 \]

\[ f_X(3) = \frac{\binom{4}{3} \cdot \binom{6}{3}}{\binom{10}{3}} = \frac{20}{120} = 0.16666... \]

The value of \( E(X) \) is:

\[ 0 \cdot f_X(0) + 1 \cdot f_X(1) + 2 \cdot f_X(2) + 3 \cdot f_X(3) = 0 + 0.3 + 2 \cdot 0.5 + 3 \cdot 0.16666... = 1.8 \]

This means that if you were to randomly continue to choose committees adhering to these rules, you’d average 1.8 males per committee.

(Two points each for each correct \( f_X \) value - i.e., two points for getting \( f_X(0) \), two points for getting \( f_X(1) \), and so on; the final two points are for getting a correct value of \( E(X) \))