Notes of the (pre-midterm) Discussion Section of Nov 15, 2002

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1 Normal Distribution

We covered Problems (1.b) and (1.c) from Midterm 2 of Fall 2001.

2 Expectation, Variance, Joint distribution

We covered Problem 4 from the Final of Fall 2001. However, the numeric calculation in the Solution has a bug. (The formula is correct.)

$$Var(Y) = E(Y^2) - (E(Y))^2 = (\sum_{t \in \text{image}(Y)} t^2 f_Y(t)) - (E(Y))^2 =$$

$$= 0 \times \frac{1}{12} + 1 \times \frac{1}{3} + 2^2 \times \frac{5}{12} + 3^2 \times \frac{1}{6} - (\frac{5}{3})^2 = \frac{42}{12} - \frac{25}{9} = \frac{126}{36} - \frac{100}{36} = \frac{26}{36} = \frac{13}{18}$$

3 The Fibonacci Numbers

Notes from “Discrete Mathematics with Applications” of Susanna Epp.

In 1202 Fibonacci, the greatest mathematician of the middle Ages, posed the following problem (God knows why...)

A single pair of rabbits (male and female) exists at year 0. Assume the following conditions:

1. Rabbit pairs are not fertile during their first year of life but thereafter give birth to one new male/female pair in every year;
2. no rabbits die.

How many rabbit pairs $F_n$ will there be at the (end of the) $n$-th year?

**Solution** The best way to solve this problem is by using recursion. Assume you know how many rabbit pairs there were at years $k - 1, k - 2, k - 3, \ldots, 2, 1, 0$. How many there will be at the $k$-th year? The crucial observation is that the number of rabbit pairs born in year $k$ is the same as the number of pairs alive in year $k - 2$. Why? Because it is exactly the rabbit pairs that were alive at year $k - 2$ that gave birth during year $k$. The rabbits born in year $k - 1$ did not give birth. So, the number of pairs of rabbits alive in year $k$ equals the ones alive at year $k - 1$ plus the pairs newly born in year $k$. Let us define

$$F_n = \begin{cases} \text{the number of rabbit pairs} \\ \text{alive in year } n \end{cases}$$

Then
\[
F_k = \begin{bmatrix}
\text{the number of rabbit pairs} \\
\text{alive in year } k
\end{bmatrix}
+ \begin{bmatrix}
\text{the number of rabbit pairs} \\
\text{born in year } k
\end{bmatrix}
+ \begin{bmatrix}
\text{the number of rabbit pairs} \\
\text{alive in year } k - 1
\end{bmatrix}
+ \begin{bmatrix}
\text{the number of rabbit pairs} \\
\text{alive in year } k - 2
\end{bmatrix}
= F_{k-1} + F_{k-2}
\]

It is \(F_0 = 1\) and \(F_1 = 1\).

The above is a second-order linear recurrence relation. Its characteristic equation

\[x^2 - x - 1 = 0\]

has two distinct solutions

\[r_1 = \frac{1 + \sqrt{5}}{2}\]
\[r_2 = \frac{1 - \sqrt{5}}{2}\]

Hence it will be

\[F_n = C \left( \frac{1 + \sqrt{5}}{2} \right)^n + D \left( \frac{1 - \sqrt{5}}{2} \right)^n\]

In order to find \(C\) and \(D\) we use the following two equations:

\[1 = F_0 = C + D\]
\[1 = F_1 = C \left( \frac{1 + \sqrt{5}}{2} \right) + D \left( \frac{1 - \sqrt{5}}{2} \right)\]

from which we derive

\[C = \frac{1 + \sqrt{5}}{2\sqrt{5}}\]
\[D = -\left( \frac{1 - \sqrt{5}}{2\sqrt{5}} \right)\]

Remark: Notice that the above recurrence relation is the same with the one that we came up with for the “tower” problem, which we covered in class. You can find the details of the “tower” problem in the set of “Final Practice Problems” (Problem 16).

### 3.1 A slight modification

Consider the following slight modification of the problem:

A single pair of rabbits (male and female) exists at year 0. Assume the following conditions:

1. Rabbit pairs are not fertile during their first year of life but thereafter give birth to two new male/female pairs in every year;
2. no rabbits die.

How many rabbit pairs \(G_n\) will there be at the (end of the) \(n\)-th year?
Solution  By following the same logic we have

\[ G_k = G_{k-1} + 2G_{k-2} \]

with \( G_0 = G_1 = 1 \).

The characteristic equation

\[ x^2 - x - 2 = 0 \]

has two distinct solutions

\[ r_1 = 2 \]

\[ r_2 = -1 \]

Hence it will be

\[ G_n = C2^n + D(-1)^n \]

We know that

\[ 1 = G_0 = C + D \]
\[ 1 = G_1 = 2C - D \]

from which we derive \( C = \frac{2}{3} \) and \( D = \frac{1}{3} \).