I. Overview of terms covered this week...

A. Probability

i. When talking about probabilities we need to define two sets:
   a. Universal Set, \( U \) - The universal set is the set of all possible outcomes. It is often referred to as the sample space.
   b. Event Set, \( E \) - To determine the probability of an event, we must define what possible outcomes the event set contains. For example, if we are rolling a six-sided die, the universal set is: \( U = \{1, 2, 3, 4, 5, 6\} \). If we are interested in the times the die roll is less than 3, the event set is \( E = \{1, 2\} \).

ii. The probability of an event occurring is defined as a function \( P : U \to \mathbb{R} \). For any event \( e \subseteq U \), \( 0 \leq P(e) \leq 1.0 \). Furthermore, \( \sum_{t \in U} P(t) = 1 \).

iii. If all of the outcomes in \( U \) are equally probable, \( U \) is said to be equiprobable. For equiprobable universal sets, \( P(E) = \frac{|E|}{|U|} \), where \( E \subseteq U \).

B. Functions

i. A function maps a domain set to a range set: \( f : A \to B \).

ii. A function \( f \) is said to be a surjection if every element in \( B \) is mapped to (also known as onto).

iii. A function \( f \) is said to be an injection if \( \forall x, y \in A, \text{ if } f(x) = f(y) \text{ then } x = y \) (also known as one-to-one).

iv. A function \( f \) is said to be a bijection if \( f \) is both an injection and surjection.

v. The inverse is a function denoted \( f^{-1} \) that maps the range to the domain. That is, if \( f : A \to B \) then \( f^{-1} : B \to A \); furthermore, \( \forall x \in A, f(f^{-1}(x)) = x \).

vi. Given a function \( f : A \to B \) and a function \( g : C \to D \), if the set of elements mapped to in \( B \) is a superset of \( C \) then the composition of \( f \) and \( g \) is written as \( gf(x) \) and is equivalent to \( g(f(x)) \).

II. Solving Simple Probability Problems

A. When asked to solve a probability problem for an equiprobable sample space, we can simply enumerate the outcomes in \( U \) and \( E \) and divide by this number.

B. EXAMPLE: Imagine it has rained for 15 of the past 100 days, while it has been sunny for the other 85 of the past 100 days. If a student is asked to pick a day from the past 100 days at random, what is the probability that she’ll pick a date when it rained?

\[ U = \{1..100\} \text{ (the past 100 days)} \]
\[ |U| = 100 \]
\[ E = \{ r_1, r_2, ..., r_{15} \} \text{ (} r_i \text{ is the date of the } i^{th} \text{ day it rained)} \]
\[ |E| = 15 \]
\[ P(E) = \frac{|E|}{|U|} = \frac{15}{100} = 0.15 \]

C. An event can be expressed as a number of event sets with various set operators applied. For example, we may be interested in the probability of the event \( E \), which is defined as \( E = A \cup B \), or \( E = A^c \cap (B \cup C) \), and so on.

D. The Principle of Include and Exclusion states that \( P(A \cup B) = P(A) + P(B) - P(A \cap B) \). This principle, along with the aid of Venn diagrams, can be used to calculate the correct equation for a more complicated event.

III. Hypergeometric Probabilities

A. If we are dealt five cards from random from a 52-card deck, what is the probability that we’ll receive a pair of Jacks? This problem can be solved using the techniques discussed earlier. Since the sample space is equiprobable, we can simply compute the probability by dividing the size of the event set by the size of the universal set. But what are the universal and event sets?

\[ U = \{ \text{all possible five card hands} \} \]

So \( U \) would contain values like \( \{2H, 4D, KD, 8S, 3D\} \)

\[ |U| = \binom{52}{5} \]

\[ E = \{ \text{all possible five card hands with two Jacks} \} \]

So \( E \) would contain values like \( \{JH, JC, 4D, 3C, 9S\} \)

\[ |E| = \binom{4}{2} \cdot \binom{48}{3} \]

\[ P = \frac{\binom{4}{2} \cdot \binom{48}{3}}{\binom{52}{5}} \]

B. For many problems of this nature we can think of the event set as \( E(C, J, c, j) \), where \( C \) is the total number of cards to choose from (i.e., 52), \( J \) is the total number of Jacks in the deck (i.e., 4), \( c \) are the cards we are dealt (i.e., 5), and \( j \) is the number of Jacks we want (i.e., 2). The probability of such an event is often written as:

\[ P(E(C, J, c, j)) = \frac{\binom{J}{j} \cdot \binom{C-J}{c-j}}{\binom{C}{c}} \]
C. The probability function occurs often (lightbulb problems, drawing cards from a deck, etc.) and is called the *hypergeometric probability distribution*.

IV. Function Notation

A. A function can be notated in one-line form if the domain is well-ordered. If the domain is the natural numbers between 1 and 5, and the range is the natural numbers between 1 and 10, and \( f(x) = 2x \), then the one-line notation would be written as: \([2, 4, 6, 8, 10]\).

B. Two line notation lists the domain element immediately above the range element. For example, the above function written in two-line notation would be:

\[
\begin{array}{c}
1 & 2 & 3 & 4 & 5 \\
2 & 4 & 6 & 8 & 10 \\
\end{array}
\]

C. The cycle notation can be used to notate permutations. (Recall that a function \( f \) can represent a permutation over set \( A \) if \( f : A \rightarrow A \) and \( f \) is a bijection.) Given a permutation in two-line notation, it is easy to construct the cycle notation. Say we have the function \( f \) defined as follows:

\[
\begin{array}{c}
1 & 2 & 3 & 4 & 5 \\
2 & 3 & 1 & 5 & 4 \\
\end{array}
\]

We can create the cycle notation by starting and the first element in the domain. Since \( 1 \rightarrow 2 \), we add: \((1,2)\) to our cycle notation. Now, we note that \( 2 \rightarrow 3 \), so we append to \((1,2)\) the value 3, so we now have: \((1,2,3)\). We see that \( 3 \rightarrow 1 \), which closes our first cycle group. Next, we see that \( 4 \rightarrow 5 \), so we add \((4,5)\). Likewise, \( 5 \rightarrow 4 \), which closes that cycle group. Our final answer can be written as:

\[ f = (1, 2, 3)(4, 5) \]

D. Relation notation. Recall a relation is some subset of \( A \times B \). If \( f : A \rightarrow B \) then \( f \) is a relation that exhibits the property that \( \forall a \in A \), there exists exactly one tuple in \( A \times B \) of the form: \((a, x)\), for any \( x \in B \). Using our example from (A), in relation notation, \( f \) would be expressed as:

\[ f = \{(1, 2), (2, 4), (3, 6), (4, 8), (5, 10)\} \]

V. Counting the Number of Possible Functions

A. Let \( f : A \rightarrow B \). How many possible functions are there? \(|B|^{|A|}\) since for each element in \( A \) we must choose one of \( B \) values for the element in the domain to map to.
B. Let \( f : A \rightarrow B \). How many possible injections are there? Each value of \( A \) must map to a unique value of \( B \). Hence there are as many functions as there are ways to form \(|A|\)-lists from a set of size \(|B|\). Specifically: \( \frac{|B|^{|A|}}{|B|-|A|!} \). (Note that here \(|A| \leq |B|\).

C. Imagine that at the prom there are a set of boys, \( B \), and a set of girls, \( G \). Let \( R \) be a relation over \( B \) and \( G \) s.t. \( R : B \rightarrow G \). If every girl has dance partner, is \( R \) a function? If every boy has a dance partner, is \( R \) a function? What conditions must hold for \( R \) to be not only a function, but a surjection? If \( R \) is a function, can \( R \) ever not be an injection?