Practice Problem Set 1, CSE21 Winter 2005

1. Check the correct answer for the following problems. You win 4 points for every correct answer. You lose 2 points for every wrong answer.

   (a) The recurrence $a_n = 2 + a_{n-1}$ $(n \geq 1)$ with $a_0 = 2$ is satisfied by

      i. the sequence $a_n = 2n$ for $n \geq 0$
      ii. the sequence $a_n = 2(n+1)$ for $n \geq 0$
      iii. both of the above

   Solution Since we did not ask for proof in the exam, just produce the first few values and the answer becomes clear.

   The prrof requires a smart trick, which goes as follows. The recurrence can be rewritten as $a_n - a_{n-1} = 2$ $(n \geq 1)$.
   The first n iterations are:
   $a_n - a_{n-1} = 2$
   $a_{n-1} - a_{n-2} = 2$
   $\ldots$
   $a_1 - a_0 = 2$
   We add them up and get:
   $a_n - a_0 = 2 + 2 + \ldots + 2 = 2 \cdot n$. That is, $a_n = 2(n+1)$.

   (b) The sequence specified by $a_n = 3a_{n-2}$, $a_0 = 1$, $a_1 = 2$ is

      i. second-order linear recurrence
      ii. geometric
      iii. none of the above

   Solution The sequence is a second-order linear recurrence according to the definition: $a_n = ba_{n-1} + ca_{n-2}$. In this sequence, $b = 0$, $c = 3$.

   (c) Let $f(x) = x\sqrt{x}$ and $g(x) = x$. It is

i. $f(x)$ is $O(g(x))$
ii. $g(x)$ is $O(f(x))$
iii. both of the above

Solution (ii)

(d) Let $f(x) = x \log x$ and $g(x) = x^2$. It is

i. $f(x)$ is $O(g(x))$
ii. $g(x)$ is $O(f(x))$
iii. both of the above

Solution (i)

(e) Let $f(x) = x + \log x$ and $g(x) = x$. It is

i. $f(x)$ is $O(g(x))$
ii. $g(x)$ is $O(f(x))$
iii. both of the above

Solution (iii)

2. Suppose a sequence satisfies the given recurrence relation and initial conditions. Find an explicit formula for each sequence.

(a)
$$a_n = a_{n-1} + 2a_{n-2}a_0 = 6, a_1 = 3$$

(b)
$$b_n = 6b_{n-1} - 9b_{n-2}b_0 = 0, b_1 = 3$$

Solution

(a) The characteristic equation is: $x^2 - x - 2 = 0$ and its solutions are: $r_1 = 2$, $r_2 = -1$. The solutions to the equations:
$K_1 + K_2 = a_0 = 6$ and $2K_1 - K_2 = 3$ are:
$K_1 = 3$, $K_2 = 3$. The final answer is: $a_n = 3 * (2^n + (-1)^n)$.

(b) The characteristic equation is: $x^2 - 6x + 9 = 0$ and its solutions are: $r_1 = r_2 = 3$. The solutions to the equations:
$K_1 = b_0 = 0$ and $3K_1 + 3K_2 = 3$ are:
$K_1 = 0$, $K_2 = 1$. The final answer is: $b_n = n3^n$.

3. A sequence is defined recursively as follows:

$$v_k = 2v_{k-2}, \forall k \geq 2, v_0 = 1, v_1 = 2$$

(a) Use iteration to guess an explicit closed formula (a function of n) for the sequence.

(b) Use induction to prove that the formula in part(a) holds for all n.

Solution

(a) The first 9 values of n give the following iteration:

$$v_0 = 1, v_1 = 2, v_2 = 2, v_3 = 4, v_4 = 4, v_5 = 8, v_6 = 8, v_7 = 16, v_8 = 16$$

Based on the above iteration it is easy to guess the explicit closed formula for the sequence, which is:

$$v_n = \begin{cases} 2^{\frac{n+1}{2}}, & \text{if n is odd;} \\ 2^{\frac{n}{2}}, & \text{if n is even.} \end{cases}$$

(b) Using strong induction to prove that the above formula holds for all n, we proceed as follows:

i. First, we have to verify that v_n is true for all $n_0 = 0 \leq n \leq n_1 = 1$, i.e., satisfies the initial conditions, or base cases. Indeed:

$$v_0 = 2^{\frac{0}{2}} = 2^0 = 1; v_1 = 2^{\frac{1+1}{2}} = 2^1 = 2$$

ii. Our induction hypothesis is that for some $n > n_1$, v_k is true for all k, where $n_0 = 0 \leq k < n$.

iii. The induction step requires us to prove that v_n is true, for all $n \geq n_0 > 0$. Breaking the proof in two cases, we have:

A. If n is odd, then $n - 2$ is also odd, and so:

$$2v_{n-2} = 2 \cdot 2^{\frac{n-2+1}{2}} = 2 \cdot 2^{\frac{n-1}{2}} = 2^{\frac{n-1}{2}+1} = 2^{\frac{n-1+2}{2}} = 2^{\frac{n+1}{2}} = v_n$$

B. Similarly, if n is even, then $n - 2$ is also even, and so:

$$2v_{n-2} = 2 \cdot 2^{\frac{n-2}{2}} = 2^{\frac{n-2}{2}+1} = 2^{\frac{n-2+2}{2}} = 2^{\frac{n}{2}} = v_n$$

4. Which of the following graphs have an Euler circuit? Prove why or prove why not.

(a) K_4, where K_4 is the complete graph with 4 nodes.

(b) $K_{1999,1999}$, where $K_{1999,1999}$ is the complete bipartite graph with 1999 nodes on each side.

(c) $K_{1998,2000}$, where $K_{1998,2000}$ is the complete bipartite graph with 1998 nodes on the one side and 2000 on the other side.

Solution We know that if a graph G is a connected graph in which every vertex has even degree then G has an Euler circuit. We also know that a complete graph is a connected one.

(a) In the case of the complete graph K_4 with 4 nodes, every node has degree 3, so it does not have an Euler circuit.

(b) In the case of the complete bipartite graph $K_{1999,1999}$, each node on each side is adjacent to all 1999 nodes of the other side, which means that it has odd degree. So $K_{1999,1999}$ does not have an Euler circuit.

(c) In the case of the complete bipartite graph $K_{1998,2000}$, each node on the 1998 side is adjacent to all 2000 nodes of the 2000 side, which means that it has even degree. Similarly, each node on the 2000 side is adjacent to all 1998 nodes of the 1998 side, which means that it also has even degree. So $K_{1998,2000}$ does have an Euler circuit.

5. Assume that you have an infinitely large set of blocks of heights 1 and 2 inches. Imagine constructing towers by piling blocks of different heights directly on top of one another. For example, a tower of height 6 inches could be obtained using any of the following sequences:

- Six 1-inch blocks. This sequence is represented as $[1, 1, 1, 1, 1, 1]$.
- Three 2-inch blocks. This sequence is represented as $[2, 2, 2]$.
- One 1-inch block, stacked on top of one 2-inch block, stacked on top of one 1-inch block, stacked on top of one 2-inch block. This sequence is represented as $[2, 1, 2, 1]$. Note that the sequence $[1, 2, 1, 2]$ is different from the sequence $[2, 1, 2, 1]$ or the sequence $[1, 1, 2, 2]$. That is, the order is important.
- etc...

Let t_n be the number of ways to construct a tower of height n inches using blocks from the set.

(a) Find a recurrence relation for t_n.

Solution Consider a n-inch tower. There are two cases:

i. Its top block is a 1-inch block. Then there are t_{n-1} ways to build the $(n-1)$ inch tower on top of which the 1-inch block is found.

ii. Its top block is a 2-inch block. Then there are t_{n-2} ways to build the $(n-2)$ inch tower on top of which the 2-inch block is found.

Hence, it is

$$t_n = t_{n-1} + t_{n-2}$$

Let's find now the initial conditions t_1 and t_2:

i. A tower of height 1 can be built in one way only: By using a single 1-inch block.

ii. A tower of height 2 can be built in two ways: Either by stacking two 1-inch blocks, i.e., by the sequence $[1, 1]$, or by using a single 2-inch block, i.e., by the sequence $[2]$.

Hence

$$t_1 = 1, t_2 = 2$$

You may also consider t_0 and t_1 to be the initial conditions. and derive them as follows:

i. A tower of height 0 can be built in one way only: Use no block, i.e., by the empty sequence $[]$.

ii. A tower of height 1 can be built in one way only: By using a single 1-inch block.

Hence

$$t_0 = 1, t_1 = 1$$

Notice that t_n is the Fibonnaci sequence.

(b) Solve the recurrence relation t_n, i.e., derive an explicit formula for t_n.

Solution t_n is a second-order linear recurrence relation. The characteristic equation of the relation is

$$t^2 - t - 1 = 0$$

The sequence has two roots r and s:

$$r = \frac{1 + \sqrt{1 - 4(-1)}}{2} = \frac{1 + \sqrt{5}}{2}$$

and

$$s = \frac{1 + \sqrt{1 - 4(-1)}}{2} = \frac{1 - \sqrt{5}}{2}$$

Hence the sequence t_n satisfies the explicit formula

$$t_n = C(\frac{1 + \sqrt{5}}{2})^n + D(\frac{1 - \sqrt{5}}{2})^n$$

To find C and D we use the initial conditions

$$t_1 = 1 = C\frac{1 + \sqrt{5}}{2} + D\frac{1 - \sqrt{5}}{2}$$

$$t_2 = 2 = C(\frac{1 + \sqrt{5}}{2})^2 + D(\frac{1 - \sqrt{5}}{2})^2$$

By solving the above system we find

$$C = \frac{1 + \sqrt{5}}{2\sqrt{5}}, D = -\frac{1 - \sqrt{5}}{2\sqrt{5}}$$

Grading policy: There will be no partial credit for finding an explicit formula for t_n if you start with the wrong recurrence relation.[1]

6. Prove or disprove the following. Use the big-oh definition.

[1] A partial grading policy would give advantage to the ones who guess an "easy" but wrong recurrence.

(a)

$$f(x) = \Sigma_{i\in\mathbf{N},0\leq i<x} r^i \text{ is (is not) } O(r^{x+1})$$

where $r > 1$. If you have not understood the definition of f check out the following derivations of $f(1), f(\frac{7}{2})$.

$$f(2) = \Sigma_{i\in\mathbf{N},0\leq i<2} r^i = \Sigma_{i=0,1} r^i = 1 + r^1$$

$$f(\frac{7}{2}) = \Sigma_{i\in\mathbf{N},0\leq i<\frac{7}{2}} r^i = \Sigma_{i=0,1,2,3} r^i = 1 + r^1 + r^2 + r^3$$

Solution Let us choose $M = \frac{1}{r-1}$ and $x_0 = 1$. We have to prove that

$$\forall x > x_0, \Sigma_{i\in\mathbf{N},0\leq i<x} r^i < Mr^{x+1}$$

It is

$$\Sigma_{i\in\mathbf{N},0\leq i<x} r^i = \Sigma_{i\in[0,\dots,\lfloor x\rfloor]} r^i \frac{r^{\lfloor x\rfloor+1} - 1}{r-1}$$

We also know that

$$\lfloor x\rfloor \leq x \Rightarrow \lfloor x\rfloor + 1 \leq x + 1$$

Hence

$$r^{\lfloor x\rfloor+1} \leq r^{x+1} \Rightarrow \frac{r^{\lfloor x\rfloor+1}}{r-1} \leq Mr^{x+1} \Rightarrow \frac{r^{\lfloor x\rfloor+1} - 1}{r-1} < Mr^{x+1}$$

(b)

$$f(x) = \Sigma_{i\in\mathbf{N},0\leq i<x} r^{2i} \text{ is (is not) } O(r^{x+1})$$

where $r > 1$.

Solution Similarly to the problem above we can derive

$$\Sigma_{i\in\mathbf{N},0\leq i<x} r^{2i} = \Sigma_{i\in[0,\ldots,\lfloor x\rfloor]}(r^2)^i = \frac{r^{2\lfloor x\rfloor+1}-1}{r^2-1}$$

We have to show that there is no $M > 0$ and $x_0 \in R$ such that

$$\forall x > x_0, \Sigma_{i\in\mathbf{N},0\leq i<x} r^{2i} < Mr^{x+1}$$

Let us use contradiction to disprove the existence of such an M and x_0: Assume that there were an M and x_0 such that

$$\forall x > x_0, \Sigma_{i\in\mathbf{N},0\leq i<x} r^{2i} < Mr^{x+1}$$

Equivalently

$$\forall x > x_0, \frac{r^{2\lfloor x\rfloor+1}-1}{r^2-1} < Mr^{x+1} \Rightarrow$$

$$\forall x > x_0, \frac{r^{2\lfloor x\rfloor+1}-1}{(r^2-1)r^{x+1}} < M \Rightarrow$$

$$\forall x > x_0, \frac{r^{2\lfloor x\rfloor}}{r^x} < M(r^2-1) \Rightarrow$$

$$\forall x > x_0, \frac{r^{2\lfloor x\rfloor}}{r^{\lfloor x\rfloor}} < M(r^2-1)$$

The last step is justified by the inequality $r^{\lfloor x\rfloor} \leq r^x$. In summary, we have shown that if $f(x)$ is $O(r^{x+1})$ then it is the case that there is an M and an x_0 such that

$$\forall x > x_0, r^{\lfloor x\rfloor} < M(r^2-1)$$

The above does not hold. For example, consider any $x > \log_r(M(r^2-1)) + 1$. Obviously $\lfloor x\rfloor > \log_r(M(r^2-1))$. Then it is

$$r^{\lfloor x\rfloor} > M(r^2-1)$$

Hence we arrived at a contradiction.

7. Let f and g be functions from the set D of real numbers into the set of non-negative real numbers. Complete the following definition:

   We say that $f(x) = O(g(x))$ if ...

   Solution: and only if there exists a positive real number M and a real number x_0 such that for all x in the common domain of f and g

   $|f(x)| \le M|g(x)|$ whenever $x > x_0$.

8. Below, you will find five recurrences. For each, say whether

   `(A)` The sequence specified by it is arithmetic

   `(B)` The sequence specified by it is geometric

   `(C)` The recurrence is second order linear homogeneous with constant coefficients

   `(D)` None of the above.

   You do not have to justify your answer: just choose the appropriate letter `A`, `B` ,`C` or `D` for each of the following:

   (a) $a_n = na_{n-1}$ for $n \ge 1$, with $a_0 = 1$
   Solution: D

   (b) $a_n = a_{n-1} + a_{n-1}$ for $n \ge 2$ with $a_0 = 1$ and $a_1 = 2$
   Solution: B

   (c) $a_n = 5 + a_{n-1}$ for $n \ge 1$ with $a_0 = 3$
   Solution: A

   (d) $a_n = a_{n-1} - a_{n-2}$ for $n \ge 2$ with $a_0 = 4$ and $a_1 = -1$
   Solution: C

9. Provide the best Big-Oh approximation, from the set of power functions, for each of the following functions. You do *not* have to justify your answer.

   (a) $f(x) = 7x^{5/2} + 6x^3 + 8x - 5 \qquad = O(x^3)$

   (b) $f(x) = 8x \log_2 x + 9\sqrt{x} + 3x^2 \qquad = O(x^2)$

10. Draw graphs that meet the following specifications or explain why it is impossible to do so.

(a) A simple graph where all vertices have degree 2.

(b) A graph with four vertices, one of degree 2 and the rest of degree 3

Solution:

(a) Typical graphs satisfying this requirement look like "circles" with more than two nodes. For example, a graph with nodes $\{v_1, v_2, v_3\}$ and edges $\{e_1, e_2, e_3\}$ where the endpoints of e_1 are (v_1, v_2), the endpoints of e_2 are (v_2, v_3) and the endpoints of e_3 are (v_3, v_1).

(b) There is no such graph. If it were such a graph G then its total degree, i.e., the sum of the degrees of its vertices, would be $2+3+3+3 = 11$. But 11 is odd and this is impossible since the degree of a graph has to even (Corollary 11.1.2 in the textbook).

11. Below you will find a code fragment for which you are asked to:

    (a) Find the number of basic arithmetic operations (ie. additions, subtractions, multiplications and divisions) performed, as a function of n

    (b) Then find the worst-case order of the algorithm (best big Oh approximation from the set of power functions) as a function of n.

```
(1)  s := 3
(2)  m := 5 * n
(3)  i := 0
(4)  while (i < m)
(5)      k := i + 4
(6)      for j = i to k
(7)          s := s + a[j]
(8)      next j
(9)  i := i + 5
(10) end while
```

Solution:

(a) We have one arithmetic operations in each one of lines (2), (5), (7), and (9). To find the total number of arithmetic operation we'll have to find out how many times each of those lines is executed - as a function of n. Let us name these times C_2, C_5, C_7 and C_9 correspondingly. Apparently the solution is $C_2 + C + 5 + C_7 + C_9$.

- $C_2 = 1$ because the line (2) is executed only once when the program starts.
- C_5 and C_9 are equal to the number of times the loop (4-10) is executed. It is easy to see that the loop is executed n times since i starts from 0, is incremented by 5 in each round and stays less than $5n$. Hence, $C_5 = C_9 = n$.
- Line 7 is part of the inner loop (6-8). Notice that for every round of the loop (4-10) line (7) is executed exactly 5 times - in the first round of the inside loop $j = i$, in the second $j = i + 1$, ..., in the fifth $j = i + 4$. Hence, $C_7 = 5n$.

The total number is $C_2 + C + 5 + C_7 + C_9 = 1 + n + n + 5n = 7n$

(b) In the worst case[2] there are $7n + 1$ arithmetic operations. Using Lemma 9.2.3 from the textbook we deduce that the algorithm is $O(n)$.

12. Solve the following recurrence relation by finding an explicit formula: $a_n = 8a_{n-1} - 12a_{n-2}$ for $n \geq 2$ with $a_0 = 5$ and $a_1 = 18$.

Solution: The characteristic equation is

$$t^2 - 8t + 12 = 0$$

and it has two solutions

$$\begin{array}{l} t_{1,2} = \frac{8 \overset{+}{-} \sqrt{8^2 - 4 \times 12}}{2} = \frac{8 \overset{+}{-} \sqrt{16}}{2} = \frac{8 \overset{+}{-} 4}{2} = 4 \overset{+}{-} 2 \Rightarrow \\ t_1 = 6, t_2 = 2 \end{array}$$

The solution is a sequence $Ct_1^n + Dt_2^n$ such that

$$\begin{array}{l} Ct_1^0 + Dt_2^0 = a_0 \\ Ct_1^1 + Dt_2^1 = a_1 \end{array}$$

[2]Indeed, in the best-case as well.

Replacing the numbers for t_1, t_2, a_0 and a_1 we get

$$C + D = 5$$
$$6C + 2D = 18$$

$$D = 5 - C \Rightarrow 6C + 2(5 - C) = 18 \Rightarrow C = 2 \Rightarrow D = 3$$

Hence the sequence is $26^n + 32^n$.

13. We define a *double Euler circuit* for a graph G to be a circuit that goes through every vertex at least once and every edge *exactly two times*.

    (a) Does every graph that has an Euler circuit also have a double Euler circuit? Answer yes or no, and justify your answer. (Meaning, if you say yes, prove it, and if you say no, exhibit a graph G which has an Euler circuit but no double Euler circuit.)

    (b) Does every graph G that has a double Euler circuit also have an Euler circuit? Answer yes or no, and justify your answer. (Meaning, if you say yes, prove it, and if you say no, exhibit a graph G which has a double Euler circuit but no Euler circuit.)

    (c) Write down a simple condition on a graph that is necessary and sufficient for it to have a double Euler circuit. That is, a statement of the form "Graph G has a double Euler circuit if and only if ...", where you fill in the "...". Prove your statement.

Solution:

(a) Yes. Consider any graph G that has an Euler circuit C of the form $ve_i \ldots e_j v$. We can construct a double Euler circuit by "concatenating" C to itself, i.e., make a circuit C_2 of the form $Ve_i \ldots e_j ve_i \ldots e_j v$.

(b) No. Consider the following graph

$$v_1 \overset{e_1}{-} v_2$$

This graph can not have an Euler circuit (its edges have odd degree) but there is the double Euler circuit $v_1 e_1 v_2 e_1 v_1$.

(c) Graph G has a double Euler circuit if and only if it is connected.

If there is a double Euler circuit the graph is connected because any two nodes v and w are connected from the double circuit. In particular, v and w appear in a double circuit and the section of the circuit from the first occurrence of v to the next occurrence of w is a walk from v to w.

If the graph G is connected there is a double Euler circuit. The proof says that for every edge e of G draw a parallel edge e' changing e_1 to e'_1, e_2 to e'_2 and so on. Note that G' is connected and all its nodes have an even number of edges. Hence there is an Euler circuit C for G'. If you change all e' to e in C you get a double Euler circuit for G.

14. A sequence is defined recursively as follows

$$\begin{aligned} t_k &= k - t_{k-1}, k > 0 \\ t_0 &= 0 \end{aligned}$$

(a) Guess an explicit formula for the sequence. You do not have to explain how you guessed what you guessed.

(b) Use induction to prove that the formula that you guessed in (**??**) is correct.

15. Consider the sequence $a_0, a_1, a_2, \ldots$ defined by the following recurrence relation.

$$\begin{aligned} a_k &= 2a_{k-1} - a_{k-2}, k > 1 \\ a_0 &= 1 \\ a_1 &= 4 \end{aligned}$$

Find an explicit formula for the sequence. Show the steps that you follow in order to derive the answer.

16. Consider the set of nodes $V = \{a, b, c, d, e, f\}$.

(a) How many simple directed graphs can you build using V ?

(b) How many simple undirected graphs can you build using V ?

Justify your answers.