CSE 21: Discrete Mathematics

Midterm 1
Solutions

1 Recurrence Relations I

1. $y_n = y_{n-1} + y_{n-1} \cdot 10\% + y_{n-2} \cdot 5\% = 1.1y_{n-1} + 0.05y_{n-2}$
   $y_0 = 1000$, $y_1 = 1000 \cdot (1 + 10\%) = 1100$.

2. The characteristic equation is: $x^2 - 1.1x - 0.05 = 0$ and its solutions are: $r_1 = \frac{11+\sqrt{141}}{20}$, $r_2 = \frac{11-\sqrt{141}}{20}$. The solutions to the equations:
   $K_1 + K_2 = 1000$ and $\frac{11+\sqrt{141}}{20}K_1 + \frac{11-\sqrt{141}}{20}K_2 = 1100$ are:
   $K_1 = \frac{70500+5500\sqrt{141}}{141}$, $K_2 = \frac{70500-5500\sqrt{141}}{141}$.
   The final answer is: $y_n = \frac{70500+5500\sqrt{141}}{141}(\frac{11+\sqrt{141}}{20})^n + \frac{70500-5500\sqrt{141}}{141}(\frac{11-\sqrt{141}}{20})^n$.

2 Recurrence Relations II

We will use the iteration method (a.k.a guess-and-prove) to solve the recurrence. Lets compute the first few terms of the sequence: $a_0 = 1$, $a_1 = 1$, $a_2 = 2$, $a_3 = 2$, $a_4 = 4$, $a_5 = 4$, so we guess the explicit formula $a_n = 2^{\lfloor \frac{n}{2} \rfloor}$ and will prove it by induction.

1. Base case.

   We verify the correctness of the formula when $n = 1$: $a_1 = 2^{\lfloor \frac{1}{2} \rfloor} = 2^0 = 1$.

2. Induction step.

   Suppose $a_n = 2^{\lfloor \frac{n}{2} \rfloor}$ for $n = 1, 2, \cdots, k$. We prove it for $k+1$.

   First let $k+1$ be even, that is $k+1 = 2m$, for some natural number m. Note that, $k-2$ is also even because $k-2 = 2m-2 = 2(m-1)$ and by the property of the floor function $\lfloor \frac{k+1}{2} \rfloor = \frac{k+1}{2}$. Now $a_{k+1} = 2a_{k-1} = 2 \cdot 2^{\lfloor \frac{k-1}{2} \rfloor} = 2 \cdot 2^{\frac{k-1}{2}} = 2^{1+\frac{k-1}{2}} = 2^{\frac{k+1}{2}} = 2^{\lfloor \frac{k+1}{2} \rfloor}$.

   Now let $k+1$ be odd then $a_{k+1} = a_k$. Since $k+1$ is odd then k is even, therefore $a_{k+1} = a_k = 2^{\lfloor \frac{k}{2} \rfloor} = 2^{\frac{k}{2}} = 2^{\lfloor \frac{k+1}{2} \rfloor}$, because when $k+1$ is odd then $\lfloor \frac{k+1}{2} \rfloor = \frac{k}{2}$.

3 Graphs I

1. v_1, v_2.

2. There is a walk $v_0 v_1$ from v_0 to v_1.

3. v_1, v_2, v_3, v_4.

4. $v_0 v_1 v_2 v_0$.

5. $v_0 v_1 v_4 v_3 v_1 v_2 v_0$.

4 Graphs II

1. We claim that G does not have an Euler circuit.

   Let $deg_{G^k}(v)$ denote the degree of node v in graph G^k. Then $deg_G(v_3) = 1 + deg_{G'}(v_3)$, because only one new edge incident to v_3 was added to the edge set of G. By the Theorem proved in class, a graph has an Euler circuit if and only if every node has even degree. Since G' has an Euler circuit, then $deg_{G'}(v_3)$ is even, and $deg_G(v_3) = 1 + deg_{G'}(v_3) = 1 + even = odd$. Thus G has a vertex with odd degree and therefore does not have an Euler circuit.

2. We present two ways to solve the problem:

   (a) To show that G has an Euler circuit we will demonstrate that G satisfies the conditions of the theorem: "*A graph has an Euler circuit if and only if all vertices have even degree and the graph is connected*".

   We claim that the degree of vertices v_1, v_2, v_4, v_5, v_6 is even because G' has an Euler circuit and no new edges incident to those vertices are added when G is constructed. Similarly $v_7, v_9, v_{10}, v_{11}, v_{12}$ also have even degrees. By construction $deg(v_{13}) = 2$, and $deg_G(v_3) = deg_{G'}(v_3) + 2$ $deg_G(v_3) = deg_{G''}(v_8) + 2$. But the presence of Euler circuit in G' and G'' implies that $deg_{G''}(v_3) = even$ and $deg_{G''}(v_8) = even$ therefore, $deg_G(v_3) = even$ and $deg_G(v_3) = even$. Thus all vertices of G have even degree.

   Now we argue that G is connected. G' and G'' are connected. By construction (v_8, v_3) connects G' and G'' in G. Vertex v_{13} is also connected to v_3 and this shows that G is a singe connected component. Now we have shown that G satisfies both conditions of the theorem, therefore G has an Euler circuit.

(b) We claim that G has an Euler circuit and will prove it by constructing such a circuit.

We know that G' has an Euler circuit, let that be $c' = v_{i1}, v_{i2}, ..., v_{i6}, v_{i1}$. If c' does not begin and end at v_3 we will transform it to a new circuit by breaking the old one at v_3 and gluing it together at v_{i1}, now we have an Euler circuit $v_3, ..., v_3$ in G'. Similarly, let c'' be an Euler circuit in G''. If it does not begin and end at v_8 we make it so, the same way we did with c', so we have an Euler circuit $v_8, ..., v_8$ in G''.

Next we construct an Euler circuit in G. Begin at v_3 traverse all the edges $E(G')$ by following the transformed Euler circuit c' in G' finishing at v_3, go to v_{13}, then to v_8, then follow the Euler circuit in G'' finishing in v_8 and then go back to v_3. Obviously we have visited all nodes of G, $V(G) = V(G') \cup V(G'') \cup \{v_{13}\}$, and have used all edges of G, $E(G) = E(G') \cup E(G'') \cup \{(v_3, v_{13}), (v_{13}, v_8), (v8, v_3)\}$ exactly once, by construction.

5 Big-Oh

We will prove that $f(x) \in O(h(x))$:

1. $f(x) \in O(g(x))$ implies that there exist positive real constants M_1 and n_1 such that: $f(x) \leq M_1 g(x)$, for all $x > n_1$.

2. $g(x) \in O(h(x))$ implies that there exist positive real constants M_2 and n_2 such that: $g(x) \leq M_2 h(x)$, for all $x > n_2$.

3. Let $n_0 = \max\{n_1, n_2\}$ (Another correct choice is $n_0 = n_1 + n_2$, when $n_1 > 0$ and $n_2 > 0$), then $f(x) \leq M_1 g(x) \leq M_1(M_2 h(x))$, for all $x > n_0$. So we have observed n_0 and a positive real constant $M = M_1 \cdot M_2$ such that, $f(x) \leq Mh(x)$ for all $x > n_0$. Therefore $f(x) \in O(h(x))$.