# CSE21 HW #5 Solution

1. (5 Points) $P(A) = 0.5$, $P(B) = 0.6$, $P(B|A) = 0.8$. Find $P(A|B) = ?$
   **Solution** $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)} = \frac{0.8 \cdot 0.5}{0.6} = \frac{2}{3}$.

2. (10 Points)

   (a) $P(A) = \frac{1}{4}$, $P(B|A) = \frac{1}{3}$, $P(A|B) = \frac{1}{2}$. $P(A \cup B) = ?$.
   **Solution**(4 pts) $P(A \cap B) = P(B|A)P(A) = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$.
   $P(B) = \frac{P(A \cap B)}{P(A|B)} = \frac{\frac{1}{12}}{\frac{1}{2}} = \frac{1}{6}$.
   $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{4} + \frac{1}{6} - \frac{1}{12} = \frac{1}{3}$.

   (b) Given that $P(A) = p$, $P(B) = 1 - \varepsilon$. Prove

   $$\frac{p - \varepsilon}{1 - \varepsilon} \leq P(A|B) \leq \frac{p}{1 - \varepsilon}.$$

   **Solution**(6 pts)
   *Proof* $P(A|B) = \frac{P(A \cap B)}{P(B)} \leq \frac{(P(A)}{(P(B)} = \frac{p}{1-\varepsilon}$.
   $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)+P(B)-P(A \cup B)}{P(B)} = \frac{p+1-\varepsilon-P(A \cap B)}{1-\varepsilon} \geq \frac{p+1-\varepsilon-1}{1-\varepsilon} = \frac{p-\varepsilon}{1-\varepsilon}$.

3. (10 Points) Let $A$ denote the event that the marble selected is yellow, and let $B$ denote the event that the marble is not black. We want to find $P(A|B)$. By definition $P(A|B) = \frac{P(A \cup B)}{P(B)}$. $A \cup B$ is the event that the marble selected is yellow and not black, therefore $A \cup B = A$.
   $P(A|B) = \frac{P(A \cup B)}{P(B)} = \frac{P(A)}{P(B)}$. $P(A) = \frac{10}{15+10+15} = \frac{1}{4}$. $P(B) = \frac{15+10}{15+10+15} = \frac{5}{8}$. $P(A|B) = \frac{2}{5}$.

4. (10 Points) A fair die is thrown twice, independently. Let $x$ be the first score and $y$ be the second score. Let $A = \{x + y = 10\}$ and $B = \{x > y\}$. Find: $P(A|B)$ and $P(B|A)$.

   **Solution** $P(A) = \frac{3}{6 \times 6} = \frac{1}{12}$. [*There are 3 combinations of $x$ and $y$ which satisfy $x + y = 10$: $(x, y) = (4, 6)$ or $(5, 5)$ or $(6, 4)$.*]

   $P(B) = \frac{15}{36} = \frac{5}{12}$. [*There are 15 combinations of $x$ and $y$ which satisfy $x > y$: $(x, y) = (6, 1)$ or $(6, 2)$, or $(6, 3)$ or $(6, 4)$ or $(6, 5)$ or $(5, 1)$ or $(5, 2)$ or $(5, 3)$ or $(5, 4)$ or $(4, 1)$ or $(4, 2)$ or $(4, 3)$ or $(3, 1)$ or $(3, 2)$ or $(2, 1)$.*]

   $P(A \cap B) = \frac{1}{36}$. [$(x, y) = (6, 4)$.]

   $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{15}$.

   $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1}{3}$.

5. (15 Points) Two boxes are given as follows:
   Box $A$ contains 8 red balls, 2 white balls, and 2 blue balls.
   Box $B$ contains 2 red balls and 6 white balls.
   A fair die is tossed: if a 3 or 6 appears, a ball is randomly chosen from $A$; otherwise a ball is randomly chosen from $B$.

   (a) Find the probability that the ball is:
   (i) red, (ii) white, (iii) blue.
   **Solution**(9 pts) Construct the corresponding stochastic tree diagram as in Fig 1. We seek $P(R)$ (the probability that the ball is red), $P(W)$ (the probability that the ball is white), $P(B)$ (the probability that the ball is blue).

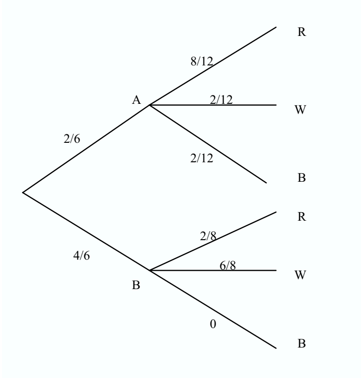


Figure 1: Problem 5 Tree Diagram

i. $P(R) = \frac{2}{6} \cdot \frac{8}{12} + \frac{4}{6} \cdot \frac{2}{8} = \frac{7}{18}$.
ii. $P(W) = \frac{2}{6} \cdot \frac{2}{12} + \frac{4}{6} \cdot \frac{6}{8} = \frac{5}{9}$.
iii. $P(B) = \frac{2}{6} \cdot \frac{2}{12} = \frac{1}{18}$.

(b) Find the probability that box $A$ was selected if the ball is red.
**Solution**(6 pts) We seek $P(A|R)$, the probability that $A$ was selected, given that the ball is red. Thus, it is necessary to find $P(A \cap R)$ and $P(R)$.
$P(A \cap R) = \frac{2}{6} \cdot \frac{8}{12} = \frac{2}{9}$.
$P(R) = \frac{7}{18}$.
$P(A|R) = \frac{P(A \cap R)}{P(R)} = \frac{4}{7}$.

6. (10 pts) There are five coins in your pocket. One has both sides painted red and one has both sides painted yellow. Each of the other three coins has one side painted red and the other side painted yellow. Now you randomly take one coin out of your pocket and toss it on a table. You see that the face that is up is red. Given this observation, what is the probability that this coin is yellow on the other side?

**Solution** Let $Y = \{$The face that is down is yellow$\}$. Let $R =\{$the face that is up is red$\}$.
$P(Y \cap R) = \frac{3}{5} \cdot \frac{1}{2} = \frac{3}{10}$.
$P(R) = \frac{3}{5} \cdot \frac{1}{2} + \frac{1}{5} = \frac{1}{2}$.
$P(Y|R) = \frac{P(Y \cap R)}{R} = \frac{3}{5}$.