Final Exam Solutions, CSE21, Fall 2001

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1 Problem One

(1) Since A is a subset of B, then whenever A occurs, B occurs. Hence, $P(B|A) = 1$. The answer is choice **3**.

(2) We use Bayes' Law, $P(A|B) = P(A \cap B)/P(B)$, which gives us

$$P(A|B) = \frac{1/4}{1/3} = \frac{3}{4}.$$

The answer is choice **1**.

(3) Now we use Inclusion-Exclusion: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. So

$$P(A \cup B) = \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}.$$

The answer is choice **2**.

(4) To find $P(A^c|B^c)$, first need to compute $P(B^c)$ and $P(A^c \cap B^c)$.

- Since, $B^c = U - B$, we find

$$P(B^c) = P(U) - P(B) = 1 - \frac{1}{3} = \frac{2}{3}.$$

- We see that $P(A^c \cap B^c)$ is the same as the universal set minus everything that is not A^c and not B^c, or rather the universal set minus everything that is A or B. So

$$P(A^c \cap B^c) = P(U - (A \cup B) = 1 - \frac{7}{12} = \frac{5}{12}.$$

- Finally, we use the definition of conditional probability and find that

$$P(A^c|B^c) = \frac{P(A^c \cap B^c)}{P(B^c)} = \frac{\frac{5}{12}}{\frac{2}{3}} = \frac{5}{8}.$$

The answer is choice **2**.

(5) We have a complete graph, so each node is adjacent to each of the other $n-1$ nodes. The degree of each node is therefore $n-1$. We know that a graph has an Euler Circuit if and only if it is connected and the degree of every vertex is even. If $n-1$ must be even, then n must be odd.

The answer is choice **2**.

2 Problem Two

- **(1)** There are 12 students total, so the teacher can choose a committee of 4 in $\binom{\mathbf{12}}{\mathbf{4}}$ ways.

- **(2)** To count the number of committees that contain at least one girl, we'll take the total number of committees and subtract the number of committees that contain no girls. This means we choose all four committee members from the nine boys, and hence there are
$$\binom{12}{4} - \binom{9}{4}$$
possible comittees containing at least one girl.

- **(3)** To find the number of committees containing exactly one girl we break the problem up into two steps.
  - First we select the one girl. There are $\binom{3}{1}$ ways to do so.
  - Next we select the remaining three committee members from the nine boys. There are $\binom{9}{3}$ ways in which to do so.

  The total number of committees containing exactly one girl are therefore
$$\binom{3}{1}\binom{9}{3}.$$

3 Problem Three

We use hypergeometric probability to solve these problems. So first, let us calculate the total number of ways to choose three lightbulbs from a set of 15. There are exactly $\binom{15}{3}$ ways to do so.

- **(1)** The probability of selecting no defective bulbs is the same as choosing all bulbs from the 10 good bulbs. The probability therefore is
$$P = \frac{\binom{10}{3}}{\binom{15}{3}}.$$

- **(2)** The probability that exactly one is defective requires that we first, choose a defective bulb. There are $\binom{5}{1}$ ways to do so. Next we must choose the remaining two bulbs from the other ten, there are $\binom{10}{2}$ ways to do so. So the probability of choosing exactly one defective bulb is

$$P = \frac{\binom{5}{1}\binom{10}{2}}{\binom{15}{3}}.$$

- **(3)** The probability that at least one is defective can be found by subtracting the probability that no bulbs are defective, from one. We calulated the probability of no defective bulbs in part one and find then

$$P = 1 - \frac{\binom{10}{3}}{\binom{15}{3}}.$$

4 Problem Four

- **(1)** Since each coin flip is independent of the next, we can multiply the probability of each flip's outcome in computing the probability of a given toss. We find

P(HHH)=$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{6}$	P(TTT)=$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{12}$
P(HHT)=$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{12}$	P(THH)=$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{6}$
P(HTH)=$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{6}$	P(THT)=$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{12}$
P(HTT)=$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{12}$	P(TTH)=$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{6}$

- **(2)** Since the random variable defined returns either zero or one, the range of X is the set $\{0, 1\}$. We now look to find $f_X(t)$, by counting the total probability of tosses that return each result in the range of X.

  - $f_X(0) = P(HHH) + P(HHT) + P(HTH) + P(HTT) = \frac{1}{6} + \frac{1}{12} + \frac{1}{6} + \frac{1}{12} = \frac{1}{2}$
  - $f_X(1) = P(THH) + P(THT) + P(TTH) + P(TTT) = \frac{1}{6} + \frac{1}{12} + \frac{1}{6} + \frac{1}{12} = \frac{1}{2}$.

- **(3)** Since Y returns the number of heads showing for a given coin toss, Y assumes values in the set $\{0, 1, 2, 3\}$. We count the number of tosses for which Y returns each value and find the distribution of Y to be

  - $f_Y(0) = P(TTT) = \frac{1}{12}$
  - $f_Y(1) = P(HTT) + P(TTH) + P(THT) = \frac{1}{12} + \frac{1}{6} + \frac{1}{12} = \frac{1}{3}$
  - $f_Y(2) = P(HTH) + P(THH) + P(HHT) = \frac{1}{6} + \frac{1}{6} + \frac{1}{12} = \frac{5}{12}$
  - $f_Y(3) = P(HHH) = \frac{1}{6}$

- **(4)** Now we wish to compute the expected value of Y. We have already calculated the probability distribution function for Y, and now we must sum over all values in the range of Y, the value itself times its respective probability. We have

$$E(Y) = \sum_{t \in image(Y)} t \cdot f_Y(t) = 0 \cdot \frac{1}{12} + 1 \cdot \frac{1}{3} + 2 \cdot \frac{5}{12} + 3 \cdot \frac{1}{6} = \mathbf{\frac{5}{3}}.$$

- **(5)** Next we want to compute the variance of Y. We do so by summing over the values in the range of Y squared, the value squared times f_Y for that value and finally subtracing the square of the mean. We get

$$Var(Y) = E(Y^2) - (E(Y))^2 = \sum_{t \in image(Y)} t^2 \cdot f_Y(t) - \mu^2 = 0 \cdot \frac{1}{12} + 1 \cdot \frac{1}{3} + 4 \cdot \frac{5}{12} + 9 \cdot \frac{1}{6} - \frac{9}{4} = 3 - \frac{9}{4} = \mathbf{\frac{5}{4}}.$$

- **(6)** Finally, we wish to compute the joint distribution, $h_{X,Y}$ of X and Y.

$h_{X,Y}$	$Y=0$	$Y=1$	$Y=2$	$Y=3$
$X=0$	0	$P(HTT)$	$P(HTH)+P(HHT)$	$P(HHH)$
$X=1$	$P(TTT)$	$P(THT)+P(TTH)$	$P(THH)$	0

Calculating these sums, we have:

$h_{X,Y}$	$Y=0$	$Y=1$	$Y=2$	$Y=3$
$X=0$	0	1/12	1/4	1/6
$X=1$	1/12	1/4	1/6	0

5 Problem Five

(1) To guess an explicit formula for this sequence, first we calculate out a few terms in the sequence.

$$\begin{aligned} c_1 &= 1 \\ c_2 &= 3c_1 + 1 = 3 \cdot 1 + 1 = 3 + 1 \\ c_3 &= 3c_2 + 1 = 3 \cdot (3+1) + 1 = 3^2 + 3 + 1 \\ c_4 &= 3c_3 + 1 = 3 \cdot (3^2 + 3 + 1) + 1 = 3^3 + 3^2 + 3 + 1 \end{aligned}$$

We guess an explicit formula for the sequence to be $c_n = 3^{n-1} + 3^{n-2} + ... + 3^2 + 3 + 1$. We have the following formula for the sum of a geometric sequence

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}.$$

So with $r = 3$ and the nth term given by the sum over the first $n-1$ powers of three, we have the following closed form

$$c_n = \frac{3^n - 1}{3 - 1} = \mathbf{\frac{3^n - 1}{2}}.$$

(2) Now we wish to prove that this closed formula is correct using induction. We establish that it holds for the base case, where $n = 1$. We are given above, that $c_n = 1$, and using our formula, we have

$$c_1 = \frac{3^1 - 2}{2} = \frac{2}{2} = 1.$$

So our hypothesis holds for $n = 1$. We assume now that it holds for all values up to n and wish to show that it holds for $n+1$ as well. Using our formula,

$$c_{n+1} = \frac{3^{n+1}-1}{2} = \frac{3 \cdot 3^n - 1}{2} = \frac{3 \cdot 3^n - 3 + 2}{2} = \frac{3 \cdot (3^n - 1)}{2} + 1 = 3 \cdot c_n + 1.$$

Which is verified by our recursion above. Hence we've proved our explicit formula for c_n.

6 Problem Six

We wish to find an explicit formula for a second order linear homogeneous recurrence relation with constant coefficients. We will do so by first finding the following characteristic equation of a_n,

$$t^k = 2t^{k-1} + 3t^{k-2}.$$

Dividing through by t^{k-2} on both sides, we have

$$t^2 - 2t^k - 3 = 0,$$

which can be factored as $(t-3)(t+1)$. Hence the roots of our characteristic equation are 3 and -1.

We know that an explicit formula for this recurrence relation is of the form

$$a_n = C \cdot 3^n + D \cdot (-1)^n.$$

We'll use our initial conditions to solve for the constants C and D.

$$a_0 = C \cdot 3^0 + D \cdot (-1)^0 = C + D = 1$$
$$a_1 = C \cdot 3^1 + D \cdot (-1)^1 = 3C - D = 2$$
$$D = 1 - C$$
$$3C - (1 - C) = 2$$
$$4C - 1 = 2$$
$$C = \frac{3}{4}$$
$$D = \frac{1}{4}$$

Our final answer is therefore

$$a_n = \frac{3}{4}3^n + \frac{1}{4}(-1)^n.$$

7 Problem Seven

This problem is done by using the tree diagrams shown in class. We want to find the probability of each event that leads to our drawing two marbles of the same color, and then sum the probabilities of all such events.

- First let's look at the probability of selecting two reds with the first marble having come from urn A. First we select urn A with probability $1/2$. Next we select a red marble from urn A with probability $3/5$. Finally, we select a second red marble from urn B, which now contains 3 marbles (since we place the one we drew from A in B) and hence this probability is $3/8$. So the probability of drawing two reds when the first marble is drawn from urn A is $\mathbf{\frac{1}{2} \cdot \frac{3}{5} \cdot \frac{3}{8}}$.
- Using similar logic, we find the probability of selecting two whites, having chosen the first white from urn A to be $\mathbf{\frac{1}{2} \cdot \frac{2}{5} \cdot \frac{3}{4}}$.
- We find the probability of selecting two reds, having chosen the first red from urn B to be $\mathbf{\frac{1}{2} \cdot \frac{2}{7} \cdot \frac{2}{3}}$.
- We find the probability of selecting two whites, having chosen the first white from urn B to be $\mathbf{\frac{1}{2} \cdot \frac{5}{7} \cdot \frac{1}{2}}$.
- Finally, we add these probabilities and find that the probability of selecting two marbles of the same color is

$$\frac{1}{2} \cdot \frac{3}{5} \cdot \frac{3}{8} + \frac{1}{2} \cdot \frac{2}{5} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{2}{7} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{5}{7} \cdot \frac{1}{2} = \mathbf{\frac{9}{80}} + \mathbf{\frac{3}{20}} + \mathbf{\frac{2}{21}} + \mathbf{\frac{5}{28}}.$$

8 Problem Eight

- **(1)** We are looking for the probability A - that the cards were drawn from box A, given B - that the card drawn is even. We would like to use Bayes' Law to find this conditional probability.
  - First, we need to find $P(A \cap B)$, the probability that we have selected from A and that the card is even. The probability of selecting from A is $1/2$. The probability of selecting an even card from box A is the number of ways to choose an even card, which is 4 since there are 4 of them, divided by the total number of cards. So,

    $$P(A \cap B) = \frac{1}{2} \cdot \frac{4}{9} = \frac{2}{9}.$$
  - Next we need to calculate $P(B)$, the probability that we draw a card and it is even. This will be the sum of the probabilities of drawing an even card from box A and of drawing an even card from box B. We've already calculated the first half. The probability of drawing an even card from box B is again, the probability

that we've selected box B, which is $1/2$, multiplied by the probability of drawing an even card from box B which is $2/5$. So,

$$P(B) = \frac{1}{2} \cdot \frac{4}{9} + \frac{1}{2} \cdot \frac{2}{5} = \frac{19}{45}.$$

– Finally we can use Bayes' Law. Putting the above probabilities together, we have

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{2}{9}}{\frac{19}{45}} = \mathbf{\frac{10}{19}}.$$

- **(2a)** We need to consider the probability of all events which lead to our drawing three even cards. Since only box A contains at least three even cards (box B has only two), we must select from box A only. The probability of selecting box A is $\frac{1}{2}$. Once we've selected box A, the probability of selecting three even cards can be found using hypergeometric probability. There are 4 even cards in box A, so the probability of selecting three and all three being even is

$$P = \frac{\binom{4}{3}}{\binom{9}{3}}.$$

Hence the probability that we choose three cards from one of the boxes and that all three are even is

$$P = \frac{1}{2}\frac{\binom{4}{3}}{\binom{9}{3}}.$$

- **(2b)** If we select three cards and find that all three are even, then we must have selected from box A, since box B does not contain three even cards. Hence the probability is **1** that we have selected from box A.

9 Problem Nine

An Euler circuit is a path that starts and ends at the same node, contains every edge in the graph, and uses each vertex at least once. We know that a graph will have an Euler circuit if and only if the degree of each node is even and it is a connected graph.

We are asked to look at complete bipartite graphs, where each node in one set $V = \{v_1, v_2, \ldots v_m\}$ is connected to each node in a set $W = \{w_1, w_2, \ldots w_n\}$. There are no edges between nodes within the same set. The first condition is satisfied, our graph is connected.

Now we need m and n such that the degree of every node is even to ensure that we will have an Euler circuit. The degree of each node in V is n, since each node in V bears and edge to each node in W, and there are n nodes in W. So n must be even. Similarly, the degree of each node in W is m, since each node in W bears and edge to each node in V and there are m nodes in V. So m must be even.

Therefore, for even values m and n a complete bipartitie graph will have an Euler circuit.

10 Problem Ten

We are told that a coin is weighted such that heads is three times as likely an outcome as tails, and are asked to find the probabilities, $P(H)$ and $P(T)$. We know that

$$P(H) + P(T) = 1.$$

From the constraints defined in our problem, we also have that

$$3 \cdot P(T) = P(H).$$

So solving our equations simultaneously, we find

$$3 \cdot P(T) + P(T) = 1$$

$$P(T) = \mathbf{\frac{1}{4}}.$$

And hence,

$$P(H) = 1 - \frac{1}{4} = \mathbf{\frac{3}{4}}.$$