Final Exam, CSE21, Fall 2001

Name:

To be completed by graders:

Problem	Points Scored	Total Points
1		20
2		6
3		6
4		13
5		10
6		10
7		10
8		15
9		5
10		5
Total		100

There are 100 points and you have 180 minutes. However, be careful about timing: Some problems are relatively straightforward and will quickly give you a good number of points, while others are relatively difficult and will require deep thinking. Do not waste too much time on the hard ones before you secure the easy points.

You may not use books, notes, calculators, etc.

DO NOT START UNTIL WE TELL YOU SO

Good Luck!

Problem 1 *20 points* This is a multiple choice question. In each part there are four choices. You must circle the number corresponding to the correct choice in each case. Points are awarded like this:

- Each correct answer is worth four points.
- Leaving it blank gets you zero points.
- For a wrong answer two points will be deducted.

(1) Consider a universal set U and two events A and B, such that $A \subseteq B$. Then

1. $P(B|A) = P(B)$

2. $P(B|A) = P(B)P(A)$

3. $P(B|A) = 1$

4. none of the above

(2) Let A and B be events with $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{4}$. Then $P(A|B) =$

1. $\frac{3}{4}$

2. $\frac{2}{3}$

3. cannot be determined from the given information

4. none of the above

(3) Again, A and B are events with $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{4}$. Then $P(A \cup B) =$

1. $\frac{2}{3}$

2. $\frac{7}{12}$

3. cannot be determined from the given information

4. none of the above

(4) Again, A and B are events with $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{4}$. Now we consider $A^c = U - A$ and $B^c = U - B$, where U is the universal set.[1] Then $P(A^c|B^c) =$

1. $\frac{3}{4}$

2. $\frac{5}{8}$

3. cannot be determined from the given information

4. none of the above

[1] A^c is called the complement of A.

(5) A simple undirected graph with vertex (node) set V and edge set E is *complete* if for every two vertices $u, v \in V$ there is an edge $\{u, v\} \in E$. That is,

$$E = \{(u, v) | u \in V, v \in V\}$$

Consider the simple undirected complete graph with n nodes. For what values of n the graph has an Euler circuit?

1. The graph has an Euler circuit if n is even.

2. The graph has an Euler circuit if n is odd.

3. cannot be determined from the given information

4. none of the above

Problem 2 *6 points* A class contains 9 boys and 3 girls.

1. In how many ways can the teacher choose a committee of 4?
2. How many of them will contain at least one girl?
3. How many of them will contain exactly one girl?

Problem 3 *6 points* Three light bulbs are chosen at random from 15 bulbs of which 5 are defective. Find the probability that

1. none is defective

2. exactly one is defective

3. at least one is defective

Problem 4 *13 points* Consider three coins. The first two are fair and the third has probability $\frac{2}{3}$ to turn heads and $\frac{1}{3}$ to turn tails. We toss all three coins. The universal set U contains the following eight outcomes

$$HHH, HHT, HTH, HTT, THH, THT, TTH, TTT$$

Notice that this is not an equiprobable space.

1. Compute the probability $P(s)$ of every outcome $s \in U$.
2. Let X be the random variable that takes the value 0 if the first coin turns out heads and 1 if it turns out tails. Provide the distribution of X.
3. Let Y be the random variable that denotes the number of heads which have occured when all three coins have been tossed. Provide the distribution of Y.
4. Compute $E(Y)$.
5. Compute $Var(Y)$.
6. Compute the joint distribution $h_{X,Y}$ of X and Y. Recall $h_{X,Y}(i,j) = P(X = i, Y = j)$.

Problem 5 *10 points* Consider the sequence $c_1, c_2, c_3, \ldots$ defined by the following recurrence relation

$$c_k = 3c_{k-1} + 1, k > 1$$
$$c_1 = 1$$

1. Guess an explicit formula for the sequence. You do not have to explain how you guessed what you guessed.
2. Use induction to prove your guess.

Hint: You may find useful the following formula we had seen in class:

$$a^0 + a^1 + a^2 + a^3 + \ldots + a^n = \frac{a^{n+1} - 1}{a - 1}$$

Problem 6 *10 points* Consider the sequence $a_0, a_1, a_2, \ldots$ defined by the following recurrence relation.

$$\begin{aligned} a_k &= 2a_{k-1} + 3a_{k-2}, k > 1 \\ a_0 &= 1 \\ a_1 &= 2 \end{aligned}$$

Find an explicit formula for the sequence. Show the steps that you follow in order to derive the answer.

Problem 7 *10 points* We are given two boxes, A and B.

- Box A contains 3 red and 2 white balls.
- Box B contains 2 red and 5 white balls.

Then the following process happens:

1. A box is selected at random
2. A ball is selected from the box selected in Step 1 and is put into the other box.
3. A ball is selected from the second box (i.e., the "other" box).

Find the probability that the ball selected in Step 2 and the ball selected in Step 3 are of the same color. Show the steps you followed in order to find the solution.

Problem 8 *15 points* Box A contains nine cards numbered 1 through 9, and box B contains five cards numbered 1 through 5.

1. A box is chosen at random and a card is drawn from the chosen box. If the number is even, find the probability that the card came from box A.

2. A box is chosen at random and 3 cards are drawn from the chosen box.

   (a) What is the probability that all three cards are even?

   (b) If all cards are even, find the probability that the cards came from box A.

Problem 9 *5 points* A complete bipartite graph on (m, n) vertices, denoted $K_{m,n}$ is a simple graph with vertices $v_1, v_2, \ldots, v_m$ and $w_1, w_2, \ldots, w_n$ that satisfies the following properties:

For all $i, k \in \{1, 2, \ldots, m\}$ and for all $j, l \in \{1, 2, \ldots, n\}$,

1. there is exactly one edge from each vertex v_i to each vertex w_j
2. there is not an edge from any vertex v_i to any other vertex v_k
3. there is not an edge from any vertex w_j to any other vertex w_l

Intuitively, the above say that every node of the set $\{v_1, v_2, \ldots, v_m\}$ is adjacent to every node of the set $\{w_1, w_2, \ldots, w_n\}$ and vice versa. But no node of $\{v_1, v_2, \ldots, v_m\}$ is adjacent to any other node of $\{v_1, v_2, \ldots, v_m\}$ and same applies for the set $\{w_1, w_2, \ldots, w_n\}$.

Consider the (simple undirected) complete bipartite graph with (m, n) nodes. For what values of m and n the graph has an Euler circuit? Justify your answer.

Problem 10 *5 points* A coin is weighted so that heads (H) is three times as likely to appear as tails (T). Find $P(H)$ and $P(T)$. Show the steps you follow to find the solution.