Practice Problems for the Second Midterm of CSE21, Part B

November 5, 2001

Some of the following problems have indeed been given in past midterms and final exams of CSE21. Others are problems that we wrote for the midterm review session.

1. Multiple choice problems

(a) Suppose $X$ is a random variable that takes the values 0 and 1 and $E(X) = \frac{1}{2}$.
   i. $P(X = \frac{1}{2}) = 1$
   ii. $P(X = 1) = \frac{1}{2}$
   iii. none of the above

Solution

\[ E(X) = 0 \cdot P(X = 0) + 1 \cdot P(X = 1) = P(X = 1). \]

So

\[ P(X = 1) = \frac{1}{2} \]

(b) We toss a fair coin 100 times. The probability to get heads 50 times is

i. $\frac{50}{100}$

ii. $\binom{100}{50} \left( \frac{1}{2} \right)^{50}$

iii. $\binom{100}{50} \left( \frac{1}{2} \right)^{100}$

iv. none of the above
Solution  Using the binomial the answer is

\[ b(50; 100, \frac{1}{2}) = \binom{100}{50} \left(\frac{1}{2}\right)^{50} \left(\frac{1}{2}\right)^{50} = \binom{100}{50} \left(\frac{1}{2}\right)^{100} \]

(c) \(X\) is a random variable with \(E(X) = 2\) and \(Var(X) = \frac{1}{3}\).

i. \(P(X > 2) = \frac{1}{4}\)

ii. \(P(X > 2) = \frac{1}{3}\)

iii. cannot be determined from the above

Solution  We have no enough information to determine the value of \(P(X > 2)\).

2. A player tosses three fair coins. He wins $15 if 3 heads occur, $6 if two heads occur, and $2 if 1 head occurs. If the game is to be fair, how much should he lose if no heads occur?

Solution  The sample space of tossing three coins is

\{ HHH, HHT, HTH, HTT, THH, THT, TTH, TTT \}

Let \(X\) be a random variable that denotes the money the player gets after tossing three coins. We use \(k\) to represent the money he loses if no heads occur. \(image(X) = \{k, 2, 6, 15\}\) and

\[ f_X(k) = P(X = k) = \frac{1}{8} \]

\[ f_X(2) = P(X = 2) = \frac{3}{8} \]

\[ f_X(6) = P(X = 6) = \frac{3}{8} \]

\[ f_X(15) = P(X = 15) = \frac{1}{8} \]

To be a fair game, we should have \(E(X) = 0\), that is,

\[ E(X) = \sum_{x \in image(X)} f_X(x) \cdot x = \frac{1}{8} \cdot k + \frac{3}{8} \cdot 2 + \frac{3}{8} \cdot 6 + \frac{1}{8} \cdot 15 = 0. \]

We get \(k = -39.\)
3. Let $X$ be a continuous random variable with probability density function

$$f_x(x) = \begin{cases} \frac{kx}{2}, & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

where $k$ is a constant. Determine the value of $k$.

**Solution** We know that for any probability density function $f_X$,

$$\int_{-\infty}^{+\infty} f_X(v)dv = 1$$

Hence $\int_{0}^{1} kxdv = 1$. It is $k = 2$.

4. A fair die is tossed. Let $X$ denote twice the number appearing, and let $Y$ denote 1 or 3 according as an odd or an even number appears. Find the distribution, expectation, variance and standard deviation of

(a) $X$, 
(b) $Y$, 
(c) $X + Y$.

**Solution** The sample space is $S = \{1, 2, 3, 4, 5, 6\}$, and each number appears with probability $\frac{1}{6}$.

(a) $X(1) = 2, X(2) = 4, X(3) = 6, X(4) = 8, X(5) = 10, X(6) = 12$. Thus $X(S) = \{2, 4, 6, 8, 10, 12\}$ and each number has probability $\frac{1}{6}$. Thus the distribution of $X$ is as shows:

<table>
<thead>
<tr>
<th>$x$</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>\frac{1}{6}</td>
<td>\frac{1}{6}</td>
<td>\frac{1}{6}</td>
<td>\frac{1}{6}</td>
<td>\frac{1}{6}</td>
<td>\frac{1}{6}</td>
</tr>
</tbody>
</table>

Accordingly,

$$\mu_X = E(X) = \sum_{x \in \text{image}(X)} xf(x) = 2 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} + 8 \cdot \frac{1}{6} + 10 \cdot \frac{1}{6} + 12 \cdot \frac{1}{6} = \frac{42}{6} = 7$$

$$E(X^2) = \sum_{x \in \text{image}(X)} x^2f(x)$$
\[
\begin{align*}
&= 4 \cdot \frac{1}{6} + 16 \cdot \frac{1}{6} + 36 \cdot \frac{1}{6} + 64 \cdot \frac{1}{6} + 100 \cdot \frac{1}{6} + 144 \cdot \frac{1}{6} = \frac{364}{6} = 60.7 \\
\sigma_X^2 &= Var(X) = E(X^2) - \mu_X^2 = 60.7 - (7)^2 = 11.7 \\
\sigma_X &= \sqrt{11.7} = 3.4
\end{align*}
\]

(b) \( Y(1) = 1, Y(2) = 3, Y(3) = 1, Y(4) = 3, Y(5) = 1, Y(6) = 3 \).

Hence \( Y(S) = \{1, 3\} \) and \( g(1) = P(Y = 1) = P(\{1, 3, 5\}) = \frac{3}{6} = \frac{1}{2} \) and \( g(3) = P(Y = 3) = P(\{2, 4, 6\}) = \frac{3}{6} = \frac{1}{2} \). Thus the distribution of \( Y \) is as follows:

| \( y \) | 1 | 3 |  \\ \\
| \( g(y) \) | \( \frac{1}{2} \) | \( \frac{1}{2} \) |

Accordingly,

\[
\begin{align*}
\mu_Y &= E(Y) = \sum_{y \in \text{image}(Y)} yg(y) = 1 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} = 2 \\
E(Y^2) &= \sum_{y \in \text{image}(Y)} y^2 g(y) = 1 \cdot \left(\frac{1}{2}\right)^2 + 9 \cdot \frac{1}{2} = 5 \\
\sigma_Y^2 &= Var(Y) = E(Y^2) - \mu_Y^2 = 5 - 2^2 = 1 \\
\sigma_Y &= \sqrt{1} = 1
\end{align*}
\]

(c) Using \( (X + Y)(s) = X(s) + Y(s) \), we obtain

\[
\begin{align*}
(X + Y)(1) &= 2 + 1 = 3 \\
(X + Y)(3) &= 6 + 1 = 7 \\
(X + Y)(5) &= 10 + 1 = 11 \\
(X + Y)(2) &= 4 + 3 = 7 \\
(X + Y)(4) &= 8 + 3 = 11 \\
(X + Y)(6) &= 12 + 3 = 15
\end{align*}
\]

Hence the image set is \( \text{image}(X + Y) = \{3, 7, 11, 15\} \) and 3 and 15 occur with probability \( \frac{1}{6} \), and 7 and 11 with probability \( \frac{2}{6} \).

That is, the distribution of \( X + Y \) is as follows:

| \( z \) | 3 | 7 | 11 | 15 |  \\ \\
| \( p(z) \) | \( \frac{1}{6} \) | \( \frac{2}{6} \) | \( \frac{2}{6} \) | \( \frac{1}{6} \) |

Thus

\[
\begin{align*}
E(X + Y) &= 3 \cdot \frac{1}{6} + 7 \cdot \frac{2}{6} + 11 \cdot \frac{2}{6} + 15 \cdot \frac{1}{6} = 9 \\
E((X + Y)^2) &= 9 \cdot \left(\frac{1}{6}\right)^2 + 49 \cdot \frac{2}{6} \cdot \frac{1}{6} + 121 \cdot \frac{2}{6} \cdot \frac{1}{6} + 225 \cdot \frac{1}{6} \cdot \frac{1}{6} = 95.7 \\
Var(X + Y) &= E((X + Y)^2) - \mu_Y^2 = 95.7 - 9^2 = 14.7 \\
\sigma_{X+Y} &= \sqrt{14.7} = 3.8
\end{align*}
\]

Observe that \( E(X) + E(Y) = 7 + 2 = 9 = E(X + Y) \), but \( Var(X) + Var(Y) = 11.7 + 1 = 12.7 \neq Var(X + Y) \).

5. A player tosses two fair coins. He wins $1 or $2 according as 1 or 2 heads appear. On the other hand, he loses $5 if no heads appear. Design a suitable random variable \( X \) and determine the expected value \( E(X) \) of the game and if it is favorable to the player.
The probability that 2 heads appear is \( \frac{1}{4} \), that 2 tails (no heads) appear is \( \frac{1}{4} \), and that 1 head appears is \( \frac{1}{2} \). Thus the probability of winning $2 is \( \frac{1}{4} \), of winning $1 is \( \frac{1}{2} \), and of losing $5 is \( \frac{1}{4} \). Hence
\[
E(X) = 2 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} - 5 \cdot \frac{1}{4} = -\frac{1}{2} = -0.25.
\]
That is, the expected value of the game is minus 25 cents, and so is unfavorable to the player. 

6. Let \( X \) be a random variable with mean \( \mu \) and standard deviation \( \sigma > 0 \); and let \( X^* \) be the standardized random variable corresponding to \( X \), i.e. \( X^* = (X - \mu)/\sigma \). Show that \( E(X^*) = 0 \) and \( Var(X^*) = 1 \). (Hence \( \sigma_{X^*} = 1 \).)

**Solution**
\[
E(X^*) = E\left(\frac{X - \mu}{\sigma}\right) = \frac{1}{\sigma}E(X - \mu) = \frac{1}{\sigma}(E(X) - \mu) = 0
\]
and
\[
Var(X^*) = Var\left(\frac{X - \mu}{\sigma}\right) = \frac{1}{\sigma^2}Var(X - \mu) = \frac{1}{\sigma^2}Var(X) = \frac{\sigma^2}{\sigma^2} = 1
\]