Homework 6 Solutions

1. A sequence is defined recursively. Use iteration to guess an explicit formula for the sequence:
   \[ d_k = 2d_{k-1} + 3, \text{ for all integers } k \geq 2 \]
   \[ d_1 = 2 \]

   **Solution:** First, we change the subscripts by one, because \( d \) starts with \( d_1 \) and not \( d_0 \). Let \( D_k = d_{k+1} \). Then:
   \[ D_k = 2D_{k-1} + 3, \text{ for all integers } k \geq 1 \]
   \[ D_0 = 2 \]

   Then we use iteration and try to guess the explicit formula for \( D \):
   \[ D_0 = 2, \ D_1 = 7, \ D_2 = 17, \ D_3 = 37, \ D_4 = 77 \]

   We observe that the difference of these numbers is a multiple of 5. More specifically, the above numbers can be rewritten as follows:
   \[ D_0 = 0 \cdot 5 + 2, \ D_1 = 1 \cdot 5 + 2, \ D_2 = 3 \cdot 5 + 2, \ D_3 = 7 \cdot 5 + 2, \ D_4 = 15 \cdot 5 + 2 \]

   We can easily see that the sequence of multiplicands 0, 1, 3, 7, 15 is described by the formula \( 2^k - 1 \). So we claim that:
   \[ D_k = (2^k - 1) \cdot 5 + 2, \text{ for all integers } k \geq 0 \]

   We need to prove that claim by induction.

   **Proof:** First, we have to verify that \( D_k \) satisfies the initial conditions, or **base cases**. Indeed:
   \[ D_0 = (2^0 - 1) \cdot 5 + 2 = 2 \]

   Then, we have to verify that \( D_k \) satisfies the recursion. Our **induction hypothesis** is that:
   \[ D_{k-1} = (2^{k-1} - 1) \cdot 5 + 2, \text{ for an integer } k > 0 \]

   and the **induction step** requires the proof of the following:
   \[ D_k = (2^k - 1) \cdot 5 + 2, \text{ for all integers } k \geq 0 \]

   The proof of the induction step is the following:
   \[ 2D_{k-1} + 3 = 2 \cdot ((2^{k-1} - 1) \cdot 5 + 2) + 3 \]
   \[ = 2 \cdot (2^{k-1} \cdot 5 - 5 + 2) + 3 \]
   \[ = 2^{k} \cdot 5 - 10 + 4 + 3 \]
   \[ = 2^{k} \cdot 5 - 5 + 2 \]
   \[ = (2^k - 1) \cdot 5 + 2 = D_k \]

   Finally, since we have \( D_k = d_{k+1} \), we have:
   \[ d_k = (2^{k-1} - 1) \cdot 5 + 2, \text{ for all integers } k \geq 1 \]
2. Suppose a sequence satisfies the given recurrence relation and initial condition. Find an explicit formula for the sequence.

\[ t_k = 6t_{k-1} - 9t_{k-2}, \text{ for all integers } k \geq 2 \]
\[ t_0 = 1 \]
\[ t_1 = 3 \]

**Solution:** The given recurrence relation is of the form \( a_n = ba_{n-1} + ca_{n-2} \) with constants \( b \) and \( c \) so we use Theorem 4 on page DT-18 of the textbook. In the case of \( t_k \) we have that:

\[ b = 6, \quad c = -9 \]

Next, according to the theorem, we form the characteristic equation of the recursion \( x^2 - bx - c = 0 \), which, in our case, is:

\[ x^2 - 6x + 9 = 0 \]

The roots \( r_1 \) and \( r_2 \) of the equation are:

\[ r_1 = r_2 = 3 \]

According to the theorem, once again, if \( r_1 = r_2 \), then the explicit formula for the sequence \( t_k \) is:

\[ t_k = K_1 \cdot r_1^k + K_2 \cdot k \cdot r_1^k, \text{ for } k \geq 0, \]

where:

\[ K_1 = t_0 = 1 \text{ and } r_1 \cdot K_1 + r_1 \cdot K_2 = t_1 = 3 \Rightarrow r_1 \cdot K_2 = 0 \Rightarrow K_2 = 0 \]

Note that \( K_2 = 0 \), because \( r_1 = 3 \neq 0 \). So the solution is

\[ t_k = 3^k, \text{ for all integers } k \geq 0 \]

3. Use mathematical induction to show that any postage of at least 8¢ can be obtained using 3¢ and 5¢ stamps.

**Solution:** We want to prove by induction the following statement:

For every \( k \geq 8 \)¢, there are integers \( a, b \geq 0 \) such that

\[ k = a \cdot (3\text{¢}) + b \cdot (5\text{¢}) \]

**Proof:** First, we have to check that the base case \( k=8 \) is true. Indeed:

\[ 8 = 1 \cdot (3\text{¢}) + 1 \cdot (5\text{¢}) = 1 \text{, by having } a = b = 1 \]

Then, we define our induction hypothesis to be:

For \( k > 8 \)¢ there are integers \( a', b' \geq 0 \), such that \( t_{k-1} = a' \cdot (3\text{¢}) + b' \cdot (5\text{¢}) \)

The induction step to be proven is the following:

For \( k \geq 8 \)¢ there are integers \( a'', b'' \geq 0 \), such that \( t_k = a'' \cdot (3\text{¢}) + b'' \cdot (5\text{¢}) \)

The proof of the induction step is split in two cases:

- If \( b' > 0 \) (i.e., there are at least one 5¢ coins), then we set:
  - \( b'' = b' - 1 \) and
  - \( a'' = a' + 2 \)

- If \( b' = 0 \), then \( a' \) must be greater or equal to 3, since \( k \geq 8 \)¢, and so we set:
  - \( b'' = 2 \) and
  - \( a'' = a' - 3 \)
4. Suppose a sequence satisfies the given recurrence relation and initial condition. Find an explicit formula for the sequence.

\[ s_k = -4s_{k-1} - 4s_{k-2}, \text{ for all integers } k \geq 2 \]
\[ s_0 = 0 \]
\[ s_1 = -1 \]

**Solution:** The given recurrence relation is of the form \( a_n = ba_{n-1} + ca_{n-2} \) with constant \( b \) and \( c \) and so we use Theorem 4 on page DT-18 of the textbook. In the case of \( s_k \) we have that:
\[ b = -4, \quad c = -4 \]

Next, according to the theorem, we form the characteristic equation of the recursion \( x^2 - bx - c = 0 \), which, in our case, is:
\[ x^2 + 4x + 4 = 0 \]

The roots \( r_1 \) and \( r_2 \) of the equation are:
\[ r_1 = r_2 = -2 \]

According to the theorem, once again, if \( r_1 = r_2 \), then the explicit formula for the sequence \( s_k \) is:
\[ s_k = K_1 \cdot r_1^k + K_2 \cdot k \cdot r_1^k, \text{ for } k \geq 0, \]
where:
\[ K_1 = s_0 = 0 \text{ and } r_1 \cdot K_1 + r_1 \cdot K_2 = s_1 = -1 \Rightarrow -2 \cdot K_2 = -1 \Rightarrow K_2 = \frac{1}{2} \]

So the solution is
\[ s_k = \frac{1}{2} \cdot k \cdot (-2)^k, \text{ for all integers } k \geq 0 \]