1 Problem One

- **Solution One, using the Rule of Product:** Since all of the books of a given size type must be placed on the shelf together, we can divide this problem into four steps:

  - **Step 1:** Count the number of ways to arrange the three blocks of books. The number of ways to arrange three distinct objects in three positions is 3! or 6.
  
  - **Step 2:** Count the number of ways to arrange the books within their respective blocks. There are 5 large books and hence 5! = 120 ways to arrange them.
  
  - **Step 3:** There are 4 medium books and hence 4! = 24 ways to arrange them.
  
  - **Step 4:** Similarly, there are 3 small books and 3! = 6 ways to arrange them.

Our final answer is the product of the number of ways to arrange the blocks of books by size (Step 1) and the number of ways to arrange the large books (Step 2), the medium books (Step 3), and the small books (Step 4):

\[
3! \cdot 5! \cdot 4! \cdot 3! = 103,680
\]

- **Solution Two, using the Rule of Sum:** We can divide the final solution set, \(S\), into parts based on which size of book occupies each space. Let

\[
S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6
\]

where the \(S_i\)'s are defined as follows:
\[ S_1 = \text{lists of large books, medium books, small books} \]
\[ S_2 = \text{lists of large books, small books, medium books} \]
\[ S_3 = \text{lists of medium books, large books, small books} \]
\[ S_4 = \text{lists of medium books, small books, large books} \]
\[ S_5 = \text{lists of small books, large books, medium books} \]
\[ S_6 = \text{lists of small books, medium books, large books} \]

In this case, each partition is actually of the same size, \(5! \cdot 4! \cdot 3! = 17,280\). When we add the sizes of \(S_1 \ldots S_6\), we find the total number of arrangements to be 103,680, the same answer as we calculated above using the Rule of Product.
2 Problem Two

- (a) We have 12 distinct people and we wish to select a subset of 4 of them. Since order does not matter, our answer is

\[ \binom{12}{4} = 495. \]

- (b) Now we need to ensure that our committee contains at least one girl. There are two ways to think about this problem.

  - **Solution 1:** We need to be careful to avoid double-counting as we count the number of committees containing at least one girl. We can do so by counting the number of committees containing exactly one, exactly two and exactly three girls. These three subsets do not overlap. So, \( S = S_1 \cup S_2 \cup S_3 \) where \( S_i \) is the number of committees consisting of exactly \( i \) girls.

    * \( |S_1| = \binom{3}{1} \cdot \binom{9}{3} \) since there are \( \binom{3}{1} \) ways to select the one girl and \( \binom{9}{3} \) ways to select the remaining three boys.
    * Similarly, \( |S_2| = \binom{3}{2} \cdot \binom{9}{2} \)
    * \( |S_3| = \binom{3}{3} \cdot \binom{9}{1} \)

    We obtain the total number of committees by summing these three quantities,

    \[ |S| = |S_1| + |S_2| + |S_3| = \binom{3}{1} \cdot \binom{9}{3} + \binom{3}{2} \cdot \binom{9}{2} + \binom{3}{3} \cdot \binom{9}{1} = 369 \]

  - **Solution 2:** Alternatively, we can count the number of committees with at least one girl by first counting the total number of possible committees, as we already did in part (a), and subtracting the number of committees that contain no girls. The number of committees containing no girls is

    \[ \binom{9}{4} \]

    as we need to select all four members from the subset of boys only. Our final answer, therefore, is

    \[ \binom{12}{4} - \binom{9}{4} = 369 \]

- (c) We’ve already solved for this in part (b). The answer is

    \[ |S_1| = \binom{3}{1} \cdot \binom{9}{3} = 252 \]
3 Problem Three

- (a) Again, let us present two solutions. Many others are also possible, depending on how you work with the Rule of Sum and the Rule of Product:

  - **Solution 1**: We can solve the problem by applying only the Rule of Product:
    * **Step 1**: There are 5 choices for \(d_3\), namely \{1, 3, 5, 7, 9\}.
    * **Step 2**: There are 8 choices for \(d_1\), namely the 9 numbers \{1, \ldots, 9\} after we exclude the number selected in Step 1.
    * **Step 3**: There are 8 choices for \(d_2\), namely the 10 numbers from \{1, \ldots, 10\} after we exclude the numbers selected in Step 1 and Step 2.

The final answer is \(5 \cdot 8 \cdot 8 = 320\).

- **Solution 2**: We want to count three digit numbers, say \(d_1d_2d_3\) where \(d_1 \neq d_2 \neq d_3\). We realize that since \(d_1\) cannot be zero, but \(d_2\) can, that the size of the set of available choices for \(d_1\) depends upon our choice for \(d_2\). To eliminate such dependency, we will break our solutions up using the Rule of Sum so that we may then apply the Rule of Product. Let \(S = S_1 \cup S_2\) where \(S_1\) is the number of solutions where \(d_2 = 0\) and \(S_2\) is the number of digits where \(d_2 \neq 0\). For the number to be odd, \(d_3\) must be one of \{1, 3, 5, 7, 9\}, hence there are five possible choices for \(d_3\).

    * In calculating \(|S_1|\), we have only 1 choice for \(d_2\), 5 choices for \(d_3\) and \((9 - 1) = 8\) choices left for \(d_1\). Therefore,
      \[|S_1| = 8 \cdot 1 \cdot 5 = 40.\]
    * By similar reasoning, in calculating \(|S_2|\), we have again 5 choices for \(d_1\), 8 choices for \(d_1\) and one less, 7 choices for \(d_2\).
      \[|S_2| = 8 \cdot 7 \cdot 5 = 280.\]

Our final answer is therefore
\[|S| = |S_1| \cup |S_2| = 40 + 280 = 320.\]

This solution was provided by one of the TAs and takes more steps. However, sometimes it’s better to be safe than sorry!

- (b) Again, there are two solutions, with different use of the Rule of Sum and the Rule of Product:

  - **Solution 1** This solution is along the same lines with **Solution 1** of (a). However, there is a complication now: The number of choices we have for \(d_1\) depends on the number of choices for \(d_3\). If \(d_3\) is 0
then the number of choices we have for $d_1$ is not reduced, since 0 was not an option for $d_1$, anyway. However, if $d_3$ is not 0, say it is 2, then there is one choice less for $d_1$ - in the particular example it cannot be 2. The situation calls for applying the Rule of Sum. Let’s split the set $S$ of all 3-digit even numbers with different digits, into $S_1$ and $S_2$ as follows:

$$S_1 = \{d_1d_2d_3 | d_3 = 0\}$$

$$S_2 = \{d_1d_2d_3 | d_3 \in \{2, 4, 6, 8\}\}$$

According to the Rule of Sum it has to be $|S| = |S_1| + |S_2|$. Now, let’s apply the Rule of Product to compute $S_1$ and $S_2$. We start with $S_1$.

* **Step 1** We have 1 choice for $d_3$
* **Step 2** We have 9 choices for $d_1$
* **Step 3** We have 8 choices for $d_2$

Hence $|S_1| = 9 \cdot 8 = 72$.

Similarly, for $S_2$

* **Step 1** We have 4 choices for $d_3$
* **Step 2** We have 8 choices for $d_1$
* **Step 3** We have 8 choices for $d_2$

Hence $|S_2| = 4 \cdot 8 \cdot 8 = 256$.

Finally, $|S| = |S_1| + |S_2| = 72 + 256 = 328$

- **Solution 2** We use the same reasoning with Solution 2 of a for even three digit numbers, however this time, both $d_2$ and $d_3$ can be zero. We need to divide our solution set $S$ into three groups. Let $S_1$ be the case where $d_3 = 0$, let $S_2$ be the case where $d_2 = 0$ and let $S_3$ be the case where neither is zero.

* We find that for $S_1$, there are 9 possible choices for $d_1$ and 8 for $d_2$. Therefore $|S_1| = 72$.

* Similarly, for $S_2$, there are four choices for $d_3$, since it must be contained in the set $\{2, 4, 6, 8\}$, and 8 choices for $d_1$. So, $|S_2| = 32$.

* Finally, for $S_3$ there are again 4 choices for $d_3$, 8 choices for $d_1$ and 7 choices for $d_2$. We have that $|S_3| = 224$.

Our final answer is the sum of these three quantities,

$$|S| = 224 + 72 + 32 = 328.$$
(c) Finally we want to count the number of numbers that are greater than 700 or divisible by 5. This is again a hard problem, because we have to avoid doublecounting/overlap. To avoid overlap, we want to count first the size of the set $S_1$ of 3-digit numbers (with different digits) greater than 700, and then the set $S_2$ of all 3-digit numbers (with different digits) divisible by five, but not greater than 700. This should ensure that we count those divisible by 5 and greater than 700 only once.

- First, we measure $|S_1|$. There are three choices for $d_1$, 9 choices for $d_2$ and 8 choices for $d_3$. Hence $|S_1| = 216$.
- Next we measure $|S_2|$. There are again, a few cases to consider. Let’s partition $S_2$ into two subsets, $S_2^1$ and $S_2^2$:
  * $S_2^1 = \{d_1d_2d_3 | d_3 = 0\}$. Then there are 6 choices for $d_1$ and 8 choices for $d_2$. Hence, $|S_2^1| = 48$.
  * $S_2^2 = \{d_1d_2d_3 | d_3 = 5\}$. There are now 5 choices for $d_1$ and again 8 choices for $d_2$. Hence, $|S_2^2| = 40$.

The total number of 3-digit numbers (with different digits) less than 700 and divisible by 5, is therefore $|S_2| = |S_2^1| + |S_2^2| = 88$.
Finally, $|S| = |S_1| + |S_2| = 216 + 88 = 304$. 

6
4 Problem Four

- (a) Again, we must choose a subset of 10 from 13 distinct items. There are \( \binom{13}{10} = 286 \) ways to do so.

- (b) If he must answer the first two, then we have 11 questions remaining, from which we must choose the other 8 questions. The number of possible combinations is \( \binom{11}{8} = 165 \).

- (c) Now he must answer number one or number two, but not both. First we count the number of ways he can select one of the first two problems - there are two. Next we count the number of ways he can answer 9 more questions. He has 11 left from which to choose, so our final answer is \( 2 \cdot \binom{11}{9} = 110 \).

- (d) If he must answer exactly 3 of the first 5 questions, we must first count how many ways he can choose 3 out of 5, which is \( \binom{5}{3} \). We multiply this quantity by the number of ways to choose the other 7 questions out of the 8 choices remaining, which is \( \binom{8}{7} \). The result is \( \binom{5}{3} \cdot \binom{8}{7} = 80 \).

- (e) Counting the number of ways he can answer at least three of the first five questions is the same as calculating the number of ways he can answer exactly three, exactly four and exactly five, and summing these numbers.

  - The number of ways he can answer exactly three, is equal to the number of ways to choose which three out of five he’ll answer, multiplied by the number of ways to select the remaining 7 from the other 8 choices. This is \( \binom{5}{3} \cdot \binom{8}{7} \).

  - Similarly, the number of ways to answer exactly four, is \( \binom{5}{4} \cdot \binom{8}{6} \) and the number of ways for him to answer exactly five of the first five is \( \binom{5}{5} \cdot \binom{8}{5} \).

Summing these we arrive at the total number of ways to answer at least three of the first five questions to be

\[
\binom{5}{3} \cdot \binom{8}{7} + \binom{5}{4} \cdot \binom{8}{6} + \binom{5}{5} \cdot \binom{8}{5} = 276
\]