Practice For the Final Exam

The following problems have been given at final exams of previous years or they are practice problems that we wrote and are are relatively close to questions you may see at the exam. Recall that the final exam will also contain material that was covered in Midterm 1 and Midterm 2. Spend some time with the midterm practice sets as well. We have also included few problems that correspond to material of Midterm 1 and Midterm 2.

**Problem 1** This is a multiple choice question. In each part there are three choices, 1, 2 or 3. You must circle the number corresponding to the best choice in each case. Points are awarded like this:

- A correct answer on any part is worth four points,
- Leaving it blank gets you zero points, but
- For a wrong answer, 2 points will be deducted, meaning there is a penalty for guessing.

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(1) The recurrence $a_n = 2 + a_{n-1}$ ($n \geq 1$) with $a_0 = 2$ is satisfied by

1. the sequence $a_n = 2n$ for $n \geq 0$
2. the sequence $a_n = 2(n + 1)$ for $n \geq 0$
3. both of the above

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**Solution** The recurrence can be rewrote as $a_n - a_{n-1} = 2$ ($n \geq 1$). The first $n$ equations are:

$a_n - a_{n-1} = 2$
\[ a_{n-1} - a_{n-2} = 2 \]
\[ \ldots \]
\[ a_1 - a_0 = 2 \]

We add them up and get:
\[ a_n - a_0 = 2 + 2 + \ldots + 2 = 2n. \] That is, \( a_n = 2(n + 1) \).

(2) Suppose \( A, B \) are events such that \( P(A \cap B) = 0.2 \) and \( P(B) = 0.8 \). Then \( P(A|B) \)

1. is 0.25
2. is 0.16
3. cannot be determined from the given information

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**Solution** \( P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.2}{0.8} = 0.25 \)

(3) Suppose \( A, B \) are independent events with \( P(A) > 0 \) and \( P(B) > 0 \) such that \( P(A|B) = 0.4 \). Then \( P(A) \)

1. is 0.4
2. is 0.2
3. cannot be determined from the given information

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**Solution** Since the events are independent

\[ P(A|B) = P(A) = 0.4 \]

You can prove the above using the definition of independence as follows:

\[ P(A \cap B) = P(A)P(B) \Rightarrow P(B)P(A|B) = P(A)P(B) \Rightarrow P(A|B) = P(A) \]
(4) Suppose $A, B$ are events such that $P(A) = 0.5$ and $P(B) = 0.2$. Then $P(A \cup B)$

1. is 0.7
2. is 0.6
3. cannot be determined from the given information

Solution  We know

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Unfortunately we do not know anything about $P(A \cap B)$ and, hence, the answer is (c).

(5) If we toss two fair dice then the probability of getting exactly one six is

1. 1/4
2. 1/6
3. neither of the above

Solution  The probability is: \( \frac{1}{6} \times \frac{5}{6} + \frac{5}{6} \times \frac{1}{6} = \frac{10}{36} \).

Problem 2  1. $P(A|B) =$

(a) $P(A) + P(B) - P(A \cap B)$
(b) $P(A) - P(B) + P(A \cap B)$
(c) $P(A \cap B)/P(B)$
Solution \quad P(A|B) = P(A \cap B)/P(B).

2. The sequence specified by \( a_n = 3a_{n-2}, \ a_0 = 1, \ a_1 = 2 \) is

(a) second-order linear recurrence  
(b) geometric  
(c) none of the above

Solution \quad The sequence is a second-order linear recurrence according to the definition: \( a_n = ba_{n-1} + ca_{n-2} \). In this sequence, \( b = 0, c = 3 \). You don’t need to find the solution of the sequence. However, here is the solution: The characteristic equation is \( x^2 - 3 = 0 \). The solution of the equation is \( r_1 = \sqrt{3}, r_2 = -\sqrt{3} \). The final answer is: \( a_n = \frac{\sqrt{3}^n}{2\sqrt{3}}((\sqrt{3} + 2) + (-1)^n(\sqrt{3} - 2)) \).

3. \( P(A_1 \cap A_2 \cap A_3) = \ldots \)

(a) \( P(A_1|A_2 \cap A_3)P(A_2|A_3)P(A_3) \)  
(b) \( P(A_1)P(A_2)P(A_3) \)  
(c) none of the above

Solution \quad \( P(A_1 \cap A_2 \cap A_3) = P(A_1|A_2 \cap A_3)P(A_2|A_3)P(A_3) \).

4. \( P(A \cup B) = P(A) + P(B) \) if and only if

(a) \( A \) and \( B \) are independent  
(b) \( A \) and \( B \) are mutually exclusive  
(c) none of the above

Solution \quad \( P(A \cup B) = P(A) + P(B) \) if and only if \( A \) and \( B \) are mutually exclusive.

5. \( A \) and \( B \) are two events. Which of the following hold?

(a) \( P(A) + P(B) \leq 1 \)  
(b) \( P(A|B) \geq P(A) \)  
(c) none of the above
Solution  The answer is none of the above because (1) It’s possible that \( P(A) + P(B) > 1 \) and (2) \( P(A) > 0 = P(A|B) \) when \( A \) and \( B \) are mutually exclusive events.

6. You roll a dice one million times. The probability of seeing two hundred thousand times “5” is

(a) close to zero
(b) \( \frac{200,000 - \frac{1,000,000}{1,000,000}}{1,000,000} \)
(c) \( \left( \frac{1}{6} \right)^{1,000,000} \)

Solution  According to the binomial formula, the probability of seeing two hundred thousand times “5” is

\[
\binom{1,000,000}{200,000} \left( \frac{1}{6} \right)^{200,000} \left( \frac{5}{6} \right)^{800,000} = \left( \frac{1,000,000}{200,000} \right) \left( \frac{5,800,000}{100,000} \right) \left( \frac{1}{6} \right)^{1000000}.
\]

7. The random variable \( X \) follows the normal distribution \( N(\mu, \sigma^2) \). Which of the following hold?

(a) \( P(X > \mu) = \frac{1}{2} \)
(b) \( P(X > \mu | X > -\mu) = \frac{1}{4} \)
(c) both of the above

Solution  Only the first statement is true.

8. Consider the random variables \( X \) and \( Y \).

(a) \( E(X + Y) = E(X) + E(Y) \), only if \( X \) and \( Y \) are independent
(b) \( E(X + Y) = E(X) + E(Y) \)
(c) none of the above

Solution  It is always the case that \( E(X + Y) = E(X) + E(Y) \).

9. Consider two independent events \( A \) and \( B \)

(a) \( P(A \cap B) = P(A) + P(B) \)
(b) \( P(A \cup B) = P(A)(1 - P(B)) + P(B) \)
(c) none of the above
Solution The answer is (b) because \( P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A)P(B) = P(A)(1 - P(B)) + P(B). \)

10. We toss a fair coin 100 times. The probability to get heads 50 times is

(a) \( \frac{50}{100} \)

(b) \( \frac{50}{100} \)

(c) none of the above

Solution According to the binomial formula, the answer is:

\[ \binom{100}{50} \frac{50}{2} = \frac{100!}{50!50!}. \]

So the final answer is (c).

Problem 3 A university offers 8 courses, numbered \( C1, C2, \ldots, C8 \). There are 40 students, and each takes three (different) courses, chosen at random from \( C1, \ldots, C8 \), independent of choices of other students.

1. What is the expected number of students enrolled in course \( C1 \)?

2. Are the events “Alice takes \( C1 \)” and “Alice takes \( C2 \)” independent? (Alice is one of the students). Prove your answer correct.

3. What is the probability that Alice and Bob have at least one course in common? (Bob is another student.)

Solution (1) The simplest way to think about the problem is to use common sense: each of 40 students takes 3 courses and there are only 8 course, on average, each course has \( \frac{40 \times 3}{8} = 15 \) students. A more “CSE21” way to reach the result is the following: Let \( X_i \) be a random variable that has value of 1 when the ith student takes \( C1 \) course, 0 otherwise. Let \( X \) be the random variable that is the number of the students taking \( C1 \). Obviously \( X = X_1 + \ldots + X_{40} \). So \( E(X) = E(X_1) + \ldots + E(X_{40}) = 40 \times \frac{3}{8} = 15 \). Notice \( E(X_i) = \frac{3}{8} \) because the probability that the ith student taking \( C1 \) is \( \frac{3}{8} \) and \( X_i \) can only have value of either 1 or 0.

(2) Let \( X \) be the event that “Alice takes \( C1 \)” and \( Y \) be the event that “Alice takes \( C2 \)”. The two events are independent if and only if \( P(X \cap Y) = P(X)P(Y) \). However, \( P(X)P(Y) = \frac{3}{8} \), \( P(X \cap Y) = \binom{5}{3} = \frac{5}{8} \). Obviously they are not equal, so the two events are not independent.
(3) The probability that Alice and Bob have no courses in common is
\[
\left( \frac{3}{5} \right) = \frac{16}{50}.
\]
So the final answer is
\[
1 - \frac{16}{50} = \frac{34}{50} = \frac{17}{25}.
\]

**Problem 4** Alice and Bob each toss a fair die, and Alice wins if the value on her die is greater than or equal to the value on Bob’s die.

1. What is the probability that Alice wins?
2. If they play 240 times, what is the expected number of games won by Alice?
3. If they play 35 times, what is the probability that Alice wins at least the first 20 games?
4. If they play 50 times, what is the probability that Alice wins exactly 30 games?

**Solution**

1. The probability that Alice wins is \(\frac{6+5+4+3+2+1}{6+6} = \frac{21}{36} = \frac{7}{12}\).
2. The answer is: \(240 \times \frac{7}{12} = 140\).
3. The answer is: \((\frac{7}{12})^{20}\).
4. The answer is: \((\frac{20}{30})^{20}(\frac{7}{12})^{20}\).

**Problem 5** Consider two boxes A and B filled with red and blue balls. Box A has 8 red and 16 blue balls. Box B has 6 red and 12 blue balls. First you choose one of the two boxes. Box A has probability 0.4 to be chosen and box B has probability 0.6 to be chosen. Let us assume that you choose at random 6 balls out of the chosen box, sampling without replacement.

What is the probability that you have picked 4 red and 2 blue balls?

Given that you have picked 4 red and 2 blue balls, what is the probability that you picked them from box A?

Consider the event “box A was chosen” and the event “4 red and 2 blue balls were picked”. Are the two events independent?

Again, you pick 6 balls from the chosen box. But now if the red balls are more than the blue balls then you pick one more ball from the other box. Let us call \(B\) the random variable that represents the total number of blue balls that you end up with. Compute its probability distribution.
Solution  (a) The probability that you have picked 4 red and 2 blue balls from the A box is: \(0.4 \times \frac{\binom{8}{4}}{\binom{10}{6}}\); the probability that you have picked 4 red and 2 blue balls from the B box is: \(0.6 \times \frac{\binom{6}{4}}{\binom{10}{6}}\). So the final answer is:

\[
0.4 \times \frac{\binom{8}{4}}{\binom{10}{6}} + 0.6 \times \frac{\binom{6}{4}}{\binom{10}{6}}.
\]

(b) Let \(Y\) be the event that you have picked 4 red and 2 blue balls, \(X\) be the event that you have picked 4 red and 2 blue balls from box A. Using the conditional probability formula: \(P(X \mid Y) = \frac{P(X \cap Y)}{P(Y)}\) when \(P(Y) \neq 0\), the solution is:

\[
P(X \mid Y) = \frac{0.4 \times \frac{\binom{8}{4}}{\binom{10}{6}}}{0.4 \times \frac{\binom{8}{4}}{\binom{10}{6}} + 0.6 \times \frac{\binom{6}{4}}{\binom{10}{6}}}.\]

(c) Events \(X\) and \(Y\) are independent if and only if \(P(X \mid Y) = P(X)\). From (b) we know the answer is NO.

(d) First, the number of blue balls you finally got can be any number from the set \(\{0, 1, 2, 3, 4, 5, 6\}\). There are only two cases where you got 0 blue ball. In the first case, you first pick 6 balls from box A and they are all red, then you pick one more from box B, it is still red. In the second case, you first pick 6 balls from box B and they are all red, then you pick one more from box A, it is still red.

\[
p(0) = 0.4 \times \frac{\binom{8}{6}}{\binom{10}{6}} \times \frac{6}{18} + 0.6 \times \frac{\binom{6}{6}}{\binom{10}{6}} \times \frac{8}{24}.
\]

There are 4 different cases where you can end up with only 1 blue ball.

- You first pick 6 balls from box A and there is only one blue ball in them, then you pick one more from box B, it is red.
- You first pick 6 balls from box B and there is only one blue ball in them, then you pick one more from box A, it is red.
- You first pick 6 balls from box A and they are all red, then you pick one more from box B, it is blue.
- You first pick 6 balls from box B and they are all red, then you pick one more from box A, it is blue.

\[
p(1) = 0.4 \times \frac{\binom{8}{5}}{\binom{10}{6}} \times \frac{6}{18} + 0.6 \times \frac{\binom{6}{5}}{\binom{10}{6}} \times \frac{8}{24} + 0.4 \times \frac{\binom{8}{5}}{\binom{10}{6}} \times \frac{12}{18} + 0.6 \times \frac{\binom{6}{5}}{\binom{10}{6}} \times \frac{16}{24}.
\]

The same reasoning applies to the computation for \(p(2), \ldots, p(6)\).
\[ p(3) = 0.4 \cdot \left( \frac{5}{9} \right)^{0.5} + 0.6 \cdot \left( \frac{5}{9} \right)^{1.2} + 0.4 \cdot \left( \frac{5}{18} \right)^{1.8} + 0.6 \cdot \left( \frac{5}{18} \right)^{1.12} + \frac{16}{21}. \]

\[ p(4) = 0.4 \cdot \left( \frac{5}{9} \right)^{1.6} + 0.6 \cdot \left( \frac{5}{9} \right)^{1.2}. \]

\[ p(5) = 0.4 \cdot \left( \frac{5}{9} \right)^{1.6} + 0.6 \cdot \left( \frac{5}{9} \right)^{1.12}. \]

\[ p(6) = 0.4 \cdot \left( \frac{5}{9} \right)^{1.6} + 0.6 \cdot \left( \frac{12}{9} \right). \]

**Problem 6** Consider a series of \( n \geq 2 \) games between Alice and Jack, in which

- The probability that Alice wins the first game is \( \frac{1}{2} \).
- The probability that she wins the second game is \( \frac{1}{2^2} \).
- In general, the probability that she wins the \( i \)th game is \( 1/2^i \), for \( i = 1, \ldots, n \).

Furthermore, the games are independent of each other.

Let us call \( X \) the random variable that represents the number of games won by Alice. Obviously, \( X \) takes any value from 0 to \( n \). Compute the following and simplify your result as much as possible.

1. \( E(X) \).
2. \( Var(X) \).

**Solution** (a) Let us call \( X_i \) the random variable that represents the result of Alice playing the \( i \)th game, that is, \( X_i = 1 \) if Alice wins the \( i \)th game, 0 if she loses. It’s easy to see that \( X = X_1 + \ldots + X_n \). So we have

\[
E(X) = E(X_1 + \ldots + X_n) = E(X_1) + \ldots + E(X_n) = \frac{1}{2} + \ldots + \frac{1}{2^n} = 1 - \frac{1}{2^n}. \]

(b) \( Var(X) = E(X^2) - E(X)^2 \)

\[
= E((X_1 + \ldots + X_n)^2) - E(X)^2 
= E(X_1^2 + \ldots + X_n^2 + \sum_{1 \leq i < j \leq n} X_i X_j) - E(X)^2 
= E(X_1^2) + \ldots + E(X_n^2) + \sum_{1 \leq i < j \leq n} E(X_i X_j) - E(X)^2 
= E(X_1) + \ldots + E(X_n) + \sum_{1 \leq i < j \leq n} E(X_i)E(X_j) - E(X)^2 
= E(X) + \sum_{1 \leq i < j \leq n} E(X_i)E(X_j) - E(X)^2 \]

\[
\frac{1}{2^6} - E(X)^2 \]
\[
E(X) + (\Sigma_{1 \leq i \leq n} \frac{1}{2^n})^2 - \Sigma_{1 \leq i \leq n} \left(\frac{1}{2^n}\right)^2 - E(X)^2 \\
= E(X) + E(X)^2 - \frac{2}{1\cdot 2} = E(X)^2 \\
= 1 - \frac{1}{2^n} + \frac{1}{3\cdot 2^{-n}} \\
\]
Notice we use the fact that all games are independent. That is the reason we can replace \(E(X_i X_j)\) with \(E(X_i)E(X_j)\).

**Problem 7** Suppose a sequence satisfies the given recurrence relation and initial conditions. Find an explicit formula for each sequence.

1.
\[
a_n = a_{n-1} + 2a_{n-2} \\
a_0 = 6, a_1 = 3
\]

2.
\[
b_n = 6b_{n-1} - 9b_{n-2} \\
b_0 = 0, b_1 = 3
\]

**Solution** (1) The characteristic equation is: \(x^2 - x - 2 = 0\) and its solutions are: \(r_1 = 2, r_2 = -1\). The solutions to the equations:
\[
K_1 + K_2 = a_0 = 6 \text{ and } 2K_1 - K_2 = 3 \text{ are:} \\
K_1 = 3, K_2 = 3. \text{ The final answer is: } a_n = 3 \cdot (2^n + (-1)^n).
\]

(2) The characteristic equation is: \(x^2 - 6x + 9 = 0\) and its solutions are:
\[
r_1 = r_2 = 3. \text{ The solutions to the equations:} \\
K_1 = a_0 = 0 \text{ and } 3K_1 + 3K_2 = 3 \text{ are:} \\
K_1 = 0, K_2 = 1. \text{ The final answer is: } b_n = n3^n.
\]

**Problem 8** A sequence is defined recursively as follows:
\[
v_k = 2v_{k-2}, \forall k \geq 2, \\
v_0 = 1, v_1 = 2
\]

1. Use iteration to guess an explicit closed formula (a function of \(n\)) for the sequence.

2. Use induction to prove that the formula in part (a) holds for all \(n\).
Solution

1. The first 9 values of $n$ give the following iteration:

$$
\begin{align*}
    v_0 &= 1 \\
    v_1 &= 2 \\
    v_2 &= 2 \\
    v_3 &= 4 \\
    v_4 &= 4 \\
    v_5 &= 8 \\
    v_6 &= 8 \\
    v_7 &= 16 \\
    v_8 &= 16
\end{align*}
$$

Based on the above iteration it’s easy to guess the explicit closed formula for the sequence, which is:

$$
v_n = \begin{cases} 
2 \frac{n+1}{2}, & \text{if } n \text{ is odd} \\
2^\frac{n}{2}, & \text{if } n \text{ is even}
\end{cases}
$$

2. Using strong induction to prove that the above formula holds for all $n$, we proceed as follows:

(a) First, we have to verify that $v_n$ is true for all $n_0 = 0 \leq n \leq n_1 = 1$, i.e., satisfies the initial conditions, or base cases. Indeed:

$$
\begin{align*}
    v_0 &= 2^\frac{0}{2} = 2^0 = 1 \\
    v_1 &= 2^\frac{1+1}{2} = 2^1 = 2
\end{align*}
$$

(b) Our induction hypothesis is that for some $n > n_1$, $v_k$ is true for all $k$, where $n_0 = 0 \leq k < n$.

(c) The induction step requires us to prove that $v_n$ is true, for all $n \geq n_0 = 0$. Breaking the proof in two cases, we have:

i. If $n$ is odd, then $n - 2$ is also odd, and so:

$$
2 \cdot v_{n-2} = 2 \cdot 2^{\frac{n-2+1}{2}} = 2 \cdot 2^{\frac{n-1}{2} - 1} = 2^{\frac{n-1}{2} + 1} = 2^{\frac{n+1}{2} - 1} = 2^{\frac{n+1}{2}} = v_n
$$
ii. Similarly, if \( n \) is even, then \( n - 2 \) is also even, and so:

\[
2 \cdot v_{n-2} = 2 \cdot 2^{\frac{n-2}{2}} = 2^{\frac{n-2+1}{2}} = 2^{\frac{n-2+2}{2}} = 2^{n} = v_n
\]

**Problem 9** Which of the following graphs have an Euler circuit? Prove why or prove why not.

- \( K_4 \), where \( K_4 \) is the complete graph with 4 nodes.
- \( K_{1999,1999} \), where \( K_{1999,1999} \) is the complete bipartite graph with 1999 nodes on each side.
- \( K_{1998,2000} \), where \( K_{1998,2000} \) is the complete bipartite graph with 1998 nodes on the one side and 2000 on the other side.

**Solution** We know that if a graph \( G \) is a connected graph in which every vertex has even degree then \( G \) has an Eulerian circuit. We also know that a complete graph is a connected one.

- In the case of the complete graph \( K_4 \) with 4 nodes, every node has degree 3, so it does not have an Eulerian circuit. Intuitively, an Eulerian circuit is a trail with sequence \( e_1, e_2, \ldots, e_n \) of distinct edges of \( G \), which uses each edge of \( G \) exactly once, for which there is a sequence \( a_1, a_2, \ldots, a_n, a_1 \) of not necessarily distinct vertices of \( G \). In the case of \( K_4 \) we cannot construct such an edge sequence without using an edge twice. Note that \( K_4 \) does not even have an Eulerian trail.

- In the case of the complete bipartite graph \( K_{1999,1999} \), each node on each side is adjacent to all 1999 nodes of the other side, which means that it has odd degree. So \( K_{1999,1999} \) neither has an Eulerian circuit nor an Eulerian trail.

- In the case of the complete bipartite graph \( K_{1998,2000} \), each node on the 1998 side is adjacent to all 2000 nodes of the 2000 side, which means that it has even degree. Similarly, each node on the 2000 side is adjacent to all 1998 nodes of the 1998 side, which means that it also has even degree. So \( K_{1998,2000} \) does have an Eulerian circuit, but not an Eulerian trail.

**Problem 10** Suppose you have 2 coins in your pocket, one fair coin (equal chance of H or T) and one fake coin with \( \frac{2}{3} \) chance of H and \( \frac{1}{3} \) chance of T. Suppose you randomly pick one coin from your pocket.
1. What is the probability the coin is fair and when you flip it you get tails.

2. What is the probability the coin is fake and when you flip it you get heads.

3. Suppose you flip the coin and heads comes up, what is the probability the coin you flipped is the fair one? The fake one?

Solution (1) By rule of product, the answer is: \( \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \).

(2) By rule of product, the answer is: \( \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3} \).

(3) Let X be the event that heads comes up, Y be the event that the coin you flipped is the fair one, Z be the event that the coin you flipped is the fake one. The question is asking you what is \( P(Y \mid X) \) and \( P(Z \mid X) \)?

\[
P(Y \mid X) = \frac{P(Y \cap X)}{P(X)} = \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}.
\]

\[
P(Z \mid X) = \frac{P(Z \cap X)}{P(X)} = \frac{\frac{1}{2} \cdot \frac{2}{3}}{\frac{1}{2}} = \frac{\frac{2}{3}}{\frac{1}{2}} = \frac{4}{3}.
\]

Problem 11 Box A contains 6 balls of which 1 is defective and box B contains 7 balls of which 3 are defective. A ball is drawn at random from each box. If one ball is defective and one is not, what is the probability that the defective one came from box A?

Solution Let X be the event that one ball is defective and one is not, Y be the event that the defective one came from box A. The question is asking you what is \( P(Y \mid X) \)?

\[
P(Y \mid X) = \frac{P(Y \cap X)}{P(X)} = \frac{\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2}}{\frac{1}{2}} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}.
\]

Problem 12 Joe tosses two dice. Let us call \( X_1 \) the random variable describing the result of tossing the first die and \( Y_1 \) the random variable describing the result of tossing the second die. Apparently the possible values for \( X_1 \) and \( Y_1 \) are 1, \ldots, 6.

1. Consider the random variable

\[
Z_1 = \min(X_1, Y_1)
\]

whose value is the minimum of the outcomes of the first die and the second die. Answer the following:

(a) Find the probability function \( f_{Z_1} \) of \( Z_1 \). Write it down in table form.
Solution

<table>
<thead>
<tr>
<th>$Z_1$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(Z_1)$</td>
<td>$\frac{11}{36}$</td>
<td>$\frac{9}{36}$</td>
<td>$\frac{7}{36}$</td>
<td>$\frac{5}{36}$</td>
<td>$\frac{3}{36}$</td>
<td>$\frac{1}{36}$</td>
</tr>
</tbody>
</table>

(b) Find the expectation $E(Z_1)$ of $Z_1$.

**Solution**

$E(Z_1) = 1 \cdot \frac{11}{36} + 2 \cdot \frac{9}{36} + 3 \cdot \frac{7}{36} + 4 \cdot \frac{5}{36} + 5 \cdot \frac{3}{36} + 6 \cdot \frac{1}{36} = \frac{91}{36}$.

(c) Find the variance $Var(Z_1)$ of $Z_1$.

**Solution**

$Var(Z_1) = E(Z_1^2) - E(Z_1)^2 = 1 \cdot \frac{11}{36} + 4 \cdot \frac{9}{36} + 9 \cdot \frac{7}{36} + 16 \cdot \frac{5}{36} + 25 \cdot \frac{3}{36} + 36 \cdot \frac{1}{36}$

2. Joe tosses the two dice another 4 times. Let us call $Z_2$ the minimum of the outcomes of the two dice the second time they were tossed, $Z_3$ the minimum of the outcomes of the two dice the third time they were tossed, and so on. Let us define the variable

$$Z = Z_1 + Z_2 + Z_3 + Z_4 + Z_5$$

Find the expectation and variance of $Z$.

**Solution**

$E(Z) = E(Z_1 + Z_2 + Z_3 + Z_4 + Z_5) = 5 \cdot \frac{91}{36} = \frac{455}{36}$

$Var(Z) = Var(Z_1 + Z_2 + Z_3 + Z_4 + Z_5) = 5 \cdot \frac{2555}{36^2 \cdot 36}$. This is because $Z_1, Z_2, Z_3, Z_4$ and $Z_5$ are all independent.

3. Are the variables $X_1$ and $Z_1$ independent? Prove why or prove why not in a mathematically sound way.

**Solution**

The variables $X_1$ and $Z_1$ are not independent. It’s should be easy to verify that the event $Z_1^{-1}(6)$ and the event $X_1^{-1}(6)$ are not independent.

4. Are the variables $Z_1$ and $Z$ independent? Prove why or prove why not in a mathematically sound way.

**Solution**

The variables $Z$ and $Z_1$ are not independent. It’s should be easy to verify that the event $Z^{-1}(5)$ and the event $Z_1^{-1}(6)$ are not independent.
Problem 13  CSE999 is taken by both CSE, Physics, and Math majors. The grade \( X_c \) of a random CSE major follows the normal distribution \( N(\mu_c, \sigma^2) \). The grade \( X_p \) of a random Physics major follows the normal distribution \( N(\mu_p, \sigma^2) \) and the grade of a random Math major is \( N(\mu_m, \sigma^2) \). 40 CSE students, 20 Math Students and 20 Physics students take the class. Finally, assume that \( \mu_p < \mu_c < \mu_m \) and in addition it is \( \mu_c = \mu_p + \epsilon \) and \( \mu_m = \mu_c + \epsilon \).

Assume that \( \mu_c = 50, \mu_p = 40, \mu_m = 60, \) and \( \sigma = 10 \). With the help of the attached standard normal curve area table find the following:

1. The probability \( P(X_c > 45) \).

   \[ \text{Solution} \quad P(X_c > 45) = \frac{1}{2} + P(50 \leq X_c < 55) = \frac{1}{2} + P(0 \leq X_c^* < \frac{55-50}{10}) = 0.5 + 0.1915 = 0.6915. \quad X_c^* \text{ is the standardized random variable corresponding to } X_c. \]

2. The probability \( P(X_p > 45) \).

   \[ \text{Solution} \quad P(X_p > 45) = P(\leq X_p^* > \frac{45-40}{10}) = 0.5 - 0.1915 = 0.3085. \quad X_p^* \text{ is the standardized random variable corresponding to } X_p. \]

3. The probability \( P(X_m < 55) \).

   \[ \text{Solution} \quad P(X_m < 55) = P(X_m^* < \frac{55-60}{10}) = P(X_m^* < -0.5) = 0.5 - 0.1915 = 0.3085. \quad X_m^* \text{ is the standardized random variable corresponding to } X_m. \]

4. The probability \( P(X_c < 55) \).

   \[ \text{Solution} \quad P(X_c < 55) = P(X_c^* < \frac{55-50}{10}) = P(X_c^* < 0.5) = 0.5 + 0.1915 = 0.6915. \quad X_c^* \text{ is the standardized random variable corresponding to } X_c. \]

5. The probability \( P(45 < X_c < 55) \).
Solution \( P(45 < X_c < 55) = P\left(\frac{45-50}{10} < X_c^* < \frac{55-50}{10}\right) = P(-0.5 < X_c^* < 0.5) = 2 \ast 0.1915 = 0.3830 \). \( X_c^* \) is the standardized random variable corresponding to \( X_c \).

6. The probability \( P(45 < X_m < 55) \).

Solution \( P(45 < X_m < 55) = P\left(\frac{45-60}{10} < X_m^* < \frac{55-60}{10}\right) = P(-1.5 < X_m^* < -0.5) = P(0.5 < X_m^* < 1.5) = 0.4332 - 0.1915 = 0.2417 \). \( X_m^* \) is the standardized random variable corresponding to \( X_m \).

7. Find the probability that the grade of a random student is greater than \( \mu_c \). Do not use the specific numbers for \( \mu_c, \mu_p, \mu_m \) and \( \sigma \).

Solution

\[
P(\text{student grade} > \mu_c) = P(X_c > \mu_c)P(CSE)+P(X_m > \mu_c)P(Math)+P(X_p > \mu_c)P(PhyS).
\]

Notice we can write

\[
P(X_m > \mu_m - \epsilon) = \frac{1}{2} + P(\mu_m < X_m < \mu_m + \epsilon)
\]

Similarly

\[
P(X_p > \mu_p - \epsilon) = \frac{1}{2} - P(\mu_p < X_p < \mu_p + \epsilon)
\]

Hence

\[
P(\text{student grade} > \mu_c) = \frac{1}{2}
\]

Problem 14 A family has 6 children.

1. Assuming that each kid may be a boy with probability \( \frac{1}{2} \) what is the probability that the family has exactly 3 boys.

2. It turns out that according to recent detailed statistics the probability of a kid being a boy is 0.498. What is the probability that the family has exactly 3 boys?
Solution  (1) The answer is: \( \left( \frac{6}{9} \right)^3 \left( \frac{2}{9} \right)^3 = \frac{1}{10}. \)

(2) The answer is: \( \left( \frac{6}{3} \right)^{0.498} \cdot 0.502^3. \)

Problem 15 There are two boxes, A and B. The first box contains 7 blue and 3 yellow marbles. The second box contains 5 blue and 5 yellow marbles. 4 marbles are selected randomly from a random box.

1. What is the probability that 3 blue and 1 yellow marble are selected?

Solution

\[
P(3b \text{ and } 1y) = P(3b \text{ and } 1y|A)P(A) + P(3b \text{ and } 1y|B)P(B) =
\]

\[
= \frac{\binom{3}{3} \binom{10}{1}}{\binom{16}{4}} \cdot \frac{1}{2} + \frac{\binom{3}{2} \binom{8}{1}}{\binom{10}{4}} \cdot \frac{1}{2} = \frac{\binom{3}{3} \binom{10}{1} + \binom{3}{2} \binom{8}{1}}{2 \binom{10}{4}}
\]

2. Assuming that 3 blue and 1 yellow marble are selected, what is the probability that the box A was selected.

Solution

\[
P(A|3b \text{ and } 1y) = \frac{P(3b \text{ and } 1y|A)P(A)}{P(3b \text{ and } 1y|A)P(A) + P(3b \text{ and } 1y|B)P(B)} =
\]

\[
= \frac{\binom{3}{3} \binom{10}{1}}{\binom{3}{3} \binom{10}{1} + \binom{3}{2} \binom{8}{1}} \cdot \frac{1}{2 \binom{10}{4}}
\]

Problem 16 Assume that you have an infinitely large set of blocks of heights 1 and 2 inches. Imagine constructing towers by piling blocks of different heights directly on top of one another. For example, a tower of height 6 inches could be obtained using any of the following sequences:

- Six 1-inch blocks. This sequence is represented as \([1,1,1,1,1,1]\).
- Three 2-inch blocks. This sequence is represented as \([2,2,2]\).
One 1-inch block, stacked on top of one 2-inch block, stacked on top of one 1-inch block, stacked on top of one 2-inch block. This sequence is represented as [2, 1, 2, 1]. Note that the sequence [1, 2, 1, 2] is different from the sequence [2, 1, 2, 1] or the sequence [1, 1, 2, 2]. That is, the order is important.

etc...

Let $t_n$ be the number of ways to construct a tower of height $n$ inches using blocks from the set.

1. Find a recurrence relation for $t_n$.

**Solution**  Consider a $n$-inch tower. There are two cases:

(a) Its top block is a 1-inch block. Then there are $t_{n-1}$ ways to build the $(n - 1)$ inch tower on top of which the 1-inch block is found.

(b) Its top block is a 2-inch block. Then there are $t_{n-2}$ ways to build the $(n - 2)$ inch tower on top of which the 2-inch block is found.

Hence, it is

$$t_n = t_{n-1} + t_{n-2}$$

Let’s find now the initial conditions $t_1$ and $t_2$:

(a) A tower of height 1 can be built in one way only: By using a single 1-inch block.

(b) A tower of height 2 can be built in two ways: Either by stacking two 1-inch blocks, i.e., by the sequence [1, 1], or by using a single 2-inch block, i.e., by the sequence [2].

Hence

$$t_1 = 1, t_2 = 2$$

You may also consider $t_0$ and $t_1$ to be the initial conditions, and derive them as follows:
(a) A tower of height 0 can be built in one way only: Use no block, i.e., by the empty sequence \( \boxed{[]} \).

(b) A tower of height 1 can be built in one way only: By using a single 1-inch block.

Hence

\[ t_0 = 1, t_1 = 1 \]

Notice that \( t_n \) is the Fibonacci sequence.

2. Solve the recurrence relation \( t_n \), i.e., derive an explicit formula for \( t_n \).

**Solution** \( t_n \) is a second-order linear recurrence relation. The characteristic equation of the relation is

\[ t^2 - t - 1 = 0 \]

The sequence has two roots \( r \) and \( s \):

\[ r = \frac{1 + \sqrt{1 - 4(-1)}}{2} = \frac{1 + \sqrt{5}}{2} \]

and

\[ s = \frac{1 + \sqrt{1 - 4(-1)}}{2} = \frac{1 - \sqrt{5}}{2} \]

Hence the sequence \( t_n \) satisfies the explicit formula

\[ t_n = C \left( \frac{1 + \sqrt{5}}{2} \right)^n + D \left( \frac{1 - \sqrt{5}}{2} \right)^n \]

To find \( C \) and \( D \) we use the initial conditions

\[ t_1 = 1 = C \frac{1 + \sqrt{5}}{2} + D \frac{1 - \sqrt{5}}{2} \]
\[ t_2 = 2 = C \left( \frac{1 + \sqrt{5}}{2} \right)^2 + D \left( \frac{1 - \sqrt{5}}{2} \right)^2 \]

By solving the above system we find

\[ C = \frac{1 + \sqrt{5}}{2\sqrt{5}}, D = -\frac{1 - \sqrt{5}}{2\sqrt{5}} \]